ANALYSIS AND DIFFUSION APPROXIMATION OF THE G/G/R MACHINE REPAIR PROBLEM WITH WARM STANDBY SPARES

BY

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Several variations to the machine repair problem, including the incorporation of spares and a generalization of the failure mode of the operating machines and the service time distributions, are investigated. The study considers the G/G/R machine repair problem with M operating machines, S warm standby spares, and R repairmen in the repair facility under steady-state conditions. The warm-standby model generalizes the no-spare model and the cold-standby model which have been considered in the literature.

The steady-state equations are formulated as diffusion equations subject to two reflecting barriers. The approximate diffusion parameters of the diffusion equations are obtained (1) under the assumption that the input characteristics of the problem are defined only by their first two moments rather than their probability

distribution function, (2) under the assumption of heavy traffic approximation, that is, queues of failed machines in the repair stage are almost always non-empty, and (3) using well-known asymptotic results from renewal theory. Approximate formulas for the probability density functions of the number of failed machines in the G/G/R machine repair problem are obtained.

A profit model is developed in order to determine the optimal values of the number of spares and the number of repairmen simultaneously. Numerical examples are provided in which several system characteristics are computed under optimal operating conditions.

Comparative analysis between the diffusion approximation results, the simulation results, and some of the exact analytic results is made for the expected number of failed machines, the machine availability, and the operative utilization. The diffusion approximation approach is accurate enough for practical purposes and is a useful method for solving complex machine repair problems.

CHAPTER 1

INTRODUCTION

A finite source queueing problem in which there is a fixed number of machines in the system is variously called the machine repair problem, the machine interference problem, the cyclic queue problem, and when incorporating spares, called the maintenance float system.

An example of the machine repair problem (MRP) is described in the following: There are ten operating machines with three repairmen to repair the machines when they break down. Each time a machine fails, it is immediately sent to a repairman for repair and then put back into operation. The machines break down with a mean of 30 minutes and the distribution of running time is negative exponential. The failed machines must wait until a repairman is available. The distribution of repair time for each repairman is also negative exponential with a mean of 45 minutes. Typical questions that may be posed are: What is the expected number of machines in operation in the steady state and What is the probability that all machines are operating?

We consider another practical example as follows: In an automated factory in continuous production, there are ten identical automatic machines (e.g., pick and place robots) in production and several identical backup machines ready for use. Whenever a machine breaks down, a backup machine is activated if any is available. The failed

machine is immediately sent to a repair shop for repair. If all spare machines are being used and a breakdown occurs, then the system becomes short in which case there are less than ten machines. Once a machine is repaired, it then becomes a spare, unless the system is short in which case the repaired machine immediately is put back into production. Any machine breaks down according to an exponential distribution with mean breakdown rate of 5 per day. A repairman can repair a machine with a mean repair rate of 1 per day. The repairmen are assumed to cost the same, $100 per day, whether busy or idle, and the cost of downtime losses of a machine is $150 per day. Each backup machine is available on a long term basis at a cost of $50 per day. Management wants to decide how many backup machines to keep and how many repairmen to hire for the repair shop to minimize the expected system cost per day.

The description of the more general MRP with standby spares is given in Section 1.1. The machine repair problem is essentially a problem in the theory of queues with a finite population size (see, e.g., Saaty [51]). The MRP is one of the most practical problems in many queueing systems and has probably become one of the most used queueing models today. The practical problem is not only to evaluate the various measures of performance but also to decide how many machines to be allocated to the repairmen at any time, or alternatively, how many repairmen to be assigned to maintain a group of machines in order for example to maximize the expected profit or minimize the expected cost per unit time in the long run.

In this study, we consider three categories of the MRP, namely, (i) the case of no spares, (ii) the case of cold standby spares, and (iii) the case of warm standby spares. A standby component is called a

"cold standby" or an "unloaded standby" if its failure rate is zero. The standby machine is referred to as "warm standby" or a "lightly loaded standby" if its failure rate is nonzero and is less than the failure rate of an operating machine (unit). The names of unloaded standby and lightly loaded standby are taken from Gnedenko et al. [21]. In general, there are two reasons why the decision-maker wants to provide standby spares, if the system is considered to be operational when at least a given number of machines is working: (i) the systems may be required to operate continuously over long duration, that is the operational efficiency of the system may need to be increased, and (ii) the availability or reliability of the system can be improved by providing sufficient standby spares.

The queueing system can be described by the distribution of interarrival times, the distribution of service times, and the number of servers in the system. Kendall [34] proposes a notation for queueing systems which is now in common use. We use Kendall's notation to propose a symbolic classification of the queues that has been extensively utilized, and define the symbols as follows:

(1) M (for Markov) denotes interarrival times or service times with negative exponential distribution.

(2) G (for general) denotes interarrival times or service times distribution of the general type.

(3) E_K (for Erlang) denotes interarrival times or service times with K-stage Erlang distribution.

(4) D (for deterministic) denotes constant interarrival times or service times distribution.

For example, a/b/c represents a queueing system in which the symbols a, b, and c stand for the interarrival time distribution, service time distribution, and number of servers, respectively.

Palm [49] first developed a model for the simplest MRP with no spares when the distributions of repair and running times are negative exponential. Feller [15] follows Palm's work to show that the MRP can be treated as a birth and death process. Feller [15] defines such measures of performance as coefficient of loss for machines and for repairmen. He shows with a specific numerical example that assigning three repairmen to 20 machines is more efficient than assigning one repairman to 6 machines. Benson and Cox [6] define other measures of performance, namely, machine availability and operative utilization. Very often, in many industrial applications it is desired to know specific system characteristics of the MRP such as machine availability, operative utilization, coefficient of loss for machines and repairmen, and so on. The important and practical problem of many industrial processes is to evaluate system characteristics of the MRP on production, given (i) the number of machines the repairmen have to be assigned; (ii) the distribution of uptimes for the machines; and (iii) the distribution of repair time for the repairmen.

Following the pioneering works of Fry [17], Palm [49], Feller [15], and Benson and Cox [6], several variations to the MRP have been considered including the incorporation of spares, the incorporation of economic features to its operation, and a generalization of the stochastic behavior of the failure mode of the operating machines and the repair time distribution. Analytic solutions of the MRP with cold standbys was first obtained for negative exponential distributions of

repair and running times by Toft and Boothroyd [66]. A partial solution for this case has been proposed by Taylor and Jackson [63]. Ashcroft [4] first studied how many machines are assigned to the single repairman in order to minimize production costs. Cost models in the MRP with cold standbys have been studied by Hilliard [30], Gross et al. [23, 24]. Ashcroft [4] and Takacs [62] investigate the MRP with single repairman for the general distribution of repair time. The MRP with multiple repairmen has recently been investigated by Maritas and Xirokostas [44], Bunday and Scraton [9], Sztrik [60, 61], Tijms and Van Hoorn [65], Haryono and Sivazlian [27], and Sivazlian and Wang [55, 56, 57] under various assumptions regarding the distributions of repair and running times.

Many of the exact solutions to queueing systems with interarrival times or service times distribution of the general type have not been found. It is extremely difficult, if not impossible, to obtain the explicit formulas such as the steady-state probability mass function, and the mean and variance of the number of customers in the system for a non-Markovian queueing system. However, it will become particularly useful to apply diffusion theory when the coefficient of variation of the service time distribution, or interarrival time distribution, or interarrival time and service time distributions is not unity. The diffusion approximation has been widely applied to the study of more complicated queueing systems having general interarrival times (general failure time distribution) and general interdeparture times (general repair time distribution). The diffusion approximation methodology uses a continuous state process as an approximation of a complex discrete

state process and replaces discrete random variables by continuous random variables.

We now present two measures of performance, namely, the traffic intensity and the server utilization which are often used in a queueing system. In the case of a single server queue, the traffic intensity is a dimensionless quantity, defined as the ratio of mean service time and mean interarrival time, or alternatively, the ratio of arrival rate and service rate. We use λ and μ to represent the arrival rate and the service rate, respectively. Then we can write traffic intensity as λ/μ. In case of a multiple-server queue (say, R servers), the server utilization (or utilization factor) is defined as $\lambda/R\mu$ which is also dimensionless and represents the fraction of time the servers are busy.

The paper on heavy traffic queues is investigated by Kingman [39]. He states that a single server queue is in "heavy traffic" when the traffic intensity is less than but close to unity. Halachmi and Franta [25] state that a multi-server queue is in "heavy traffic" when the utilization per server is less than but close to unity. For the diffusion approximation of the MRP, the arrival process and departure process depend on each other in general. However, according to the point of view of several authors such as Heyman [28], Kleinrock [40], Halachmi and Franta [25], and Gelenbe and Pujolle [20], if the queue of failed machines in the repair stage is not empty, the independence criteria of the two processes are approximately justified. If the system has a very low probability of being empty, or if the probability of the system having at least nonzero failed machines in the repair stage is close to unity, then this system corresponds to the assumption

of heavy traffic case. On the basis of the heavy traffic conditions, Kleinrock [40] proposes that the arrival process and the departure process be both approximately independent renewal processes. When time t becomes large, these processes can be approximated by continuous normal (Gaussian) processes.

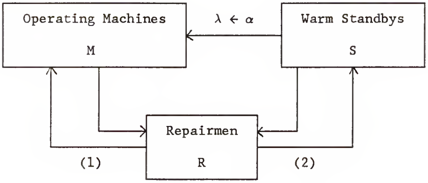
1.1 Problem Statement

We consider a model with $N = M + S$ identical machines. As many as M of these can be operating simultaneously in parallel, the rest of the S machines are warm standby spares.

We assume that warm standbys are allowed to fail while inactive before they are put into full operation, and that the spares are continuously monitored by a fault detecting device in order to identify if they are in a failed state or not. We also assume that the switchover time from standby state to operating state, from failure to repair, or from repaired to standby state (or operating state if the system is short) is instantaneous. Suppose that the failure rate of each machine (operating or standby) is independent of the state of the other machines and that the failure rates are independent of time spent in a particular state. Each of the operating machines fails independently of the state of the others with failure rate λ. Whenever one of these machines fails, it is immediately replaced by a standby if any is available. We now assume that each of the available standby spare machines fails independently of the state of all the others with failure rate α $(0 \leq \alpha \leq \lambda)$, and that when a spare moves into an operating state, its failure characteristics will be that of an operating machine. Whenever an operating machine or a spare fails it is

immediately sent to a repair facility where failed machines are repaired in the order of their breakdowns, with identical repair rate μ. Further, the succession of repair times and the succession of breakdown times are independently distributed random variables. Service is provided on a first-come first-served (FCFS) discipline. Each repairman can repair only one failed machine at a time. Suppose that the repair is independent of the failure of the operating and the spare machines, and that there are R repairmen (servers) in this repair facility. If an operating machine fails (or spare fails) and one spare is available at an instant when the repairman is available, the failed machine at once goes for repair, and the spare is put into operation. If all repairmen are busy, then a failed machine must wait until a repairman is available. Moreover, if all spares are being used and a breakdown occurs, then we say that the system becomes short in which case there are less than M operating machines. We assume that the system has M machines working in parallel and the system requires a minimum of one machine in operation to function properly. Once a machine is repaired, it is as good as new and goes into standby or operating state. When the repair of a failed machine is completed, it is then treated as a warm standby unless the system is short in which case the repaired machine is sent back immediately to an operating stage and the status of the machine is considered as good as new. It should be noted that the problem parameters λ, α, and μ will later be used to define the first moment of the MRP with general distribution using diffusion approximation in Chapter 3. The above machine repair model is shown in Figure 1.1. Note that the case of $\alpha = 0$ corresponds to spares operating

under cold-standby condition. The case when $\alpha = \lambda$ corresponds to spares operating under hot-standby condition or operating in parallel.



(1): The system is short. (2): The system is not short.
λ: The failure rate of an operating machine.
α: The failure rate of a standby machine.

Figure 1.1. The Machine Repair Model With Warm Standby Spares

Let the discrete random variable N(t) denote the number of failed machines in the system at time t and let n be the number of failed machines in the system, where n = 0, 1, ..., N. We define

(1.1) $P(n;t) = \text{Prob}[N(t) = n], \quad n = 0, 1, \ldots, N.$

We assume

$$P(n;0) = 1, \quad \text{if } n = n_0,$$

$$P(n;0) = 0, \quad \text{if } n \neq n_0 \ (n_0 = 0, 1, 2, \ldots, N).$$

Clearly, the number of failed machines in the system is described by a birth and death Markov process. This system has (N + 1) states, where state n (n = 0, 1, ..., N) corresponds to the number of failed machines which are currently in repair or are waiting for repair. The parameters of the process (λ_n, μ_n), respectively, express the birth rate (the arrival rate) and the death rate (the repair rate) in state n.

1.2 Diffusion Approximation

The method of diffusion approximation is based on the assumption that queues of failed machines in the repair stage are almost always non-empty. For each case we develop first the differential difference equations for $P(n;t)$. Under diffusion approximation, a discrete variable n is approximated by a continuous variable x, and the solutions of differential difference equations for $P(n;t)$ can be treated approximately by a set of Fokker-Planck equations for a probability density function $f(x;t)$ which in its general form is given by

$$(1.2) \qquad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x)f(x;t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(x)f(x;t)],$$

where $A_1(x)$ and $A_2(x)$ denote the respective infinitesimal mean and infinitesimal variance (or diffusion parameters) of the diffusion process $X(t)$, the number of failed machine in the system at time t, which are determined by the following equations (Cox and Miller [11]):

$$(1.3) \qquad A_1(x) = \lim_{dt \to 0} \frac{E[X(t+dt) - X(t)|X(t) = x]}{dt},$$

and

$$(1.4) \qquad A_2(x) = \lim_{dt \to 0} \frac{Var[X(t+dt) - X(t)|X(t) = x]}{dt}.$$

In our particular case, the diffusion process $\{X(t);\ t \geq 0\}$ approximates the discrete process $\{N(t);\ t \geq 0\}$, where $N(t)$ is the number of failed machines in the system at time t, and $f(x;t)$ is the probability density function of $X(t)$.

Under steady-state conditions, the approximating probability density function $f(x) = \lim_{t \to \infty} f(x;t)$ of the number of failed machines

x in the system satisfies the equation

$$(1.5) \qquad 0 = - \frac{d}{dx} [A_1(x)f(x)] + \frac{1}{2} \frac{d^2}{dx^2} [A_2(x)f(x)].$$

The exact values for the mean and variance of the number of failed machines in the steady-state can be solved easily for the M/M/R MRP, but the exact and tractable steady-state solutions are unknown for the more general G/G/R MRP. We want to replace a discrete process with the same infinitesimal mean and variance by a continuous process, and use renewal limit theorem to obtain the diffusion parameters under the assumption of heavy traffic condition. Therefore, the estimation of the diffusion parameters $A_1(x)$ and $A_2(x)$ will become tractable and will be used to set up the steady-state diffusion equations for the probability density function for the queue length, i.e., the number of failed machines in the G/G/R MRP. These equations are solved analytically to obtain approximate closed form formulas for the steady-state probability density function f(x) for the G/G/R MRP.

1.3 Boundary Condition

Since the diffusion process is restricted within the interval [0, N], we may treat x = 0 and x = N as two reflecting barriers. The existence of two reflecting barriers means that the diffusion process can not go beyond these barriers.

It will be shown later on that the density function f(x;t) has to be divided into three different ranges, namely,

$$(1.6) \qquad f(x;t) = f_i(x;t), \text{ for } x \in \Phi_i, \ i = 1, 2, 3,$$

where

$$\Phi_1 = \{x | \ 0 \le x < R\},$$

$$\Phi_2 = \{x | \ R \le x \le S\},$$

$$\Phi_3 = \{x | \ S < x \le N\}.$$

We assume that $f(x;t)$ is continuous at $x = R$ and $x = S$, and that the first and second derivatives of $f(x;t)$ with respect to x exist at $x = R^-$, $x = R^+$, $x = S^-$, and $x = S^+$.

We have three diffusion equations for $f(x;t)$ for each range of x respectively given by

$$(1.7) \qquad \frac{\partial f_i(x;t)}{\partial t} = - \frac{\partial}{\partial x} [A_{1i}(x) f_i(x;t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [A_{2i}(x) f_i(x;t)],$$

for $i = 1, 2, 3$, where $A_{1i}(x)$ and $A_{2i}(x)$ are the diffusion parameters of $X(t)$. It should be noted that $A_{1i}(x)$ and $A_{2i}(x)$ are linear functions of x, for $i = 1, 2, 3$.

Under steady-state conditions, we let

$$(1.8) \qquad f_i(x) = \lim_{t \to \infty} f_i(x;t), \quad \text{for } i = 1, 2, 3.$$

From equation 1.8, equation 1.7 will become a homogeneous second-order linear differential equation (DE) given by

$$(1.9) \qquad - \frac{d}{dx} [A_{1i}(x) f_i(x)] + \frac{1}{2} \frac{d^2}{dx^2} [A_{2i}(x) f_i(x)] = 0.$$

We know that for $0 \le x \le N$,

$$(1.10) \qquad \int_0^N f(x;t)dx = 1, \quad \text{for all } t > 0.$$

Consequently,

$$(1.11) \qquad \frac{d}{dt} \int_0^N f(x;t)dx = \int_0^N \frac{\partial f(x;t)}{\partial t} dx = 0.$$

Now from equations 1.7 and equation 1.11, we obtain

$$\int_0^N \frac{\partial f(x;t)}{\partial t}\,dx \tag{1.12}$$

$$= \int_0^R \frac{\partial f_1(x;t)}{\partial t}\,dx + \int_R^S \frac{\partial f_2(x;t)}{\partial t}\,dx + \int_S^N \frac{\partial f_3(x;t)}{\partial t}\,dx$$

$$= \int_0^R \frac{\partial}{\partial x}\{[-A_{11}(x)f_1(x;t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_{21}(x)f_1(x;t)]\}\,dx$$

$$+ \int_R^S \frac{\partial}{\partial x}\{[-A_{12}(x)f_2(x;t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_{22}(x)f_2(x;t)]\}\,dx$$

$$+ \int_S^N \frac{\partial}{\partial x}\{[-A_{13}(x)f_3(x;t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_{23}(x)f_3(x;t)]\}\,dx$$

$$= \{[-A_{11}(x)f_1(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{21}(x)f_1(x;t)]\}\Big|_{x=R}$$

$$- \{[-A_{11}(x)f_1(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{21}(x)f_1(x;t)]\}\Big|_{x=0}$$

$$+ \{[-A_{12}(x)f_2(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{22}(x)f_2(x;t)]\}\Big|_{x=S}$$

$$- \{[-A_{12}(x)f_2(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{22}(x)f_2(x;t)]\}\Big|_{x=R}$$

$$+ \{[-A_{13}(x)f_3(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{23}(x)f_3(x;t)]\}\Big|_{x=N}$$

$$- \{[-A_{13}(x)f_3(x;t)] + \frac{1}{2}\frac{\partial}{\partial x}[A_{23}(x)f_3(x;t)]\}\Big|_{x=S} = 0.$$

This equality must hold for all $t > 0$. In steady-state, we obtain

$$(1.13)\qquad \left\{\left[-A_{11}(x)f_1(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{21}(x)f_1(x)\right]\right\}\Big|_{x=R}$$

$$-\left\{\left[-A_{11}(x)f_1(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{21}(x)f_1(x)\right]\right\}\Big|_{x=0}$$

$$+\left\{\left[-A_{12}(x)f_2(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{22}(x)f_2(x)\right]\right\}\Big|_{x=S}$$

$$-\left\{\left[-A_{12}(x)f_2(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{22}(x)f_2(x)\right]\right\}\Big|_{x=R}$$

$$+\left\{\left[-A_{13}(x)f_3(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{23}(x)f_3(x)\right]\right\}\Big|_{x=N}$$

$$-\left\{\left[-A_{13}(x)f_3(x)\right] + \frac{1}{2}\frac{d}{dx}\left[A_{23}(x)f_3(x)\right]\right\}\Big|_{x=S} = 0.$$

Equation 1.13 specifies the boundary condition of the diffusion process in the steady-state for the case when $R \le S$. The boundary condition for the other case when $R > S$ is to replace respectively R by S and S by R in equation 1.13.

1.4 Solutions of the Diffusion Equations in Steady-State

The general solution of equation 1.9 is given by

$$(1.14)\qquad f_i(x) = K_i \frac{\Omega_i(x)}{A_{2i}(x)} + C_i \frac{\Omega_i(x)\psi_i(x)}{A_{2i}(x)}$$

$$\equiv G_i(x) + H_i(x), \quad \text{for } i = 1, 2, 3,$$

where K_i and C_i are positive constants,

$$(1.15)\qquad \Omega_i(x) = \exp\left\{\int^x \frac{2A_{1i}(y)}{A_{2i}(y)}\,dy\right\}, \quad x \in \Phi_i,$$

and

$$(1.16)\qquad \psi_i(x) = \int^x \frac{1}{\Omega_i(z)}\,dz, \qquad x \in \Phi_i.$$

We shall call $G_i(x) = K_i \dfrac{\Omega_i(x)}{A_{2i}(x)}$ as the first solution and

$H_i(x) = C_i \dfrac{\Omega_i(x)\psi_i(x)}{A_{2i}(x)}$ as the second solution.

There are six unknown constants K_i and C_i ($i = 1, 2, 3$) to be determined which require six conditions to be given for their specification. A condition comes from the normalization criterion $\int_0^N f(x)dx = 1$. Two other conditions are given by the continuity of $f_i(x)$ at $x = R$ and $x = S$, namely, $f_1(R) = f_2(R)$ and $f_2(S) = f_3(S)$. The other conditions are dictated by the assumed shape properties of $f(x)$, namely that $f(x)$ and $f'(x)$ vanish at either one or both of its tails. It is shown in particular that those tail conditions justify the elimination of the second solution from the general solution of the diffusion equation (i.e., $C_i = 0$ for $i = 1, 2, 3$). The K_i's are then determined from the first three conditions.

In expression 1.13 for the boundary condition, let

$$(1.17)\qquad -A_{11}(0)f_1(0) + \frac{1}{2}\frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=0} = D_1,$$

$$(1.18)\qquad -A_{11}(R)f_1(R) + \frac{1}{2}\frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=R} = D_2,$$

$$(1.19)\qquad -A_{12}(R)f_2(R) + \frac{1}{2}\frac{d}{dx}[A_{22}(x)f_2(x)]\Big|_{x=R} = D_3,$$

$$(1.20)\qquad -A_{12}(S)f_2(S) + \frac{1}{2}\frac{d}{dx}[A_{22}(x)f_2(x)]\Big|_{x=S} = D_4,$$

$$(1.21) \qquad -A_{13}(S)f_3(S) + \frac{1}{2}\frac{d}{dx}[A_{23}(x)f_3(x)]\Big|_{x=S} = D_5,$$

$$(1.22) \qquad -A_{13}(N)f_3(N) + \frac{1}{2}\frac{d}{dx}[A_{23}(x)f_3(x)]\Big|_{x=N} = D_6,$$

where D_j ($j = 1, 2, \ldots, 6$) are quantities to be determined.

Equations 1.17 through 1.22 have the following properties.

Lemma 1.1

If the general solution of equation 1.9 is given by equation 1.14, then (i) $D_1 = D_2 = 1/2\ C_1$, for $0 \le x \le R$, (ii) $D_3 = D_4 = 1/2\ C_2$, for $R \le x \le S$, and (iii) $D_5 = D_6 = 1/2\ C_3$, for $S \le x \le N$.

Proof: (i) for $0 \le x \le R$, we have

$$(1.23) \qquad f_1(x) = K_1 \frac{\Omega_1(x)}{A_{21}(x)} + C_1 \frac{\Omega_1(x)\psi_1(x)}{A_{21}(x)},$$

where K_1 and C_1 are positive constants, and

$$(1.24) \qquad \Omega_1(x) = \exp\left(\int_0^x \frac{2A_{11}(y)}{A_{21}(y)}\,dy\right), \quad \psi_1(x) = \int_0^x \frac{1}{\Omega_1(z)}\,dz.$$

Substituting $x = 0$ into equation 1.24, we have

$$(1.25) \qquad \Omega_1(0) = 1 \quad \text{and} \quad \psi_1(0) = 0.$$

Substituting $x = 0$ and $x = R$ into equation 1.23, respectively and simplifying, we obtain

$$(1.26) \qquad f_1(0) = K_1 / A_{21}(0),$$

and

$$(1.27) \qquad f_1(R) = \frac{\Omega_1(R)}{A_{21}(R)}[K_1 + C_1\psi_1(R)].$$

Multiplying $A_{21}(x)$ on both sides of equation 1.23, we have

$$(1.28) \qquad A_{21}(x)f_1(x) = K_1\Omega_1(x) + C_1\psi_1(x)\Omega_1(x).$$

Differentiating equation 1.28 with respect to x, we obtain

$$(1.29) \qquad \frac{d}{dx}[A_{21}(x)f_1(x)] = \frac{2A_{11}(x)\Omega_1(x)}{A_{21}(x)}[K_1 + C_1\psi_1(x)] + C_1.$$

Substituting $x = 0$ into equation 1.29 and simplifying, we obtain

$$(1.30) \qquad \frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=0} = \frac{2K_1A_{11}(0)}{A_{21}(0)} + C_1.$$

Substituting equations 1.26 and 1.30 into equation 1.17 and simplifying, we obtain

$$D_1 = 1/2\ C_1.$$

Similarly, substitute $x = R$ into equation 1.29 and simplify, we obtain

$$(1.31) \qquad \frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=R} = \frac{2A_{11}(R)\Omega_1(R)}{A_{21}(R)}[K_1 + C_1\psi_1(R)] + C_1.$$

Substituting equations 1.27 and 1.31 into equation 1.18 and simplifying, we obtain

$$D_2 = 1/2\ C_1.$$

Hence $D_1 = D_2 = 1/2\ C_1$.

(ii) and (iii) have similar proofs as (i).∎

Recall that

$$H_i(x) = C_i \frac{\Omega_1(x)\psi_1(x)}{A_{21}(x)}$$

$$= C_i h_i(x), \quad \text{for } i = 1, 2, 3,$$

is the second solution of equation 1.9, where $\Omega_i(x)$ and $\psi_i(x)$ are given in equations 1.15 and 1.16, respectively.

Theorem 1.1

Let equation 1.14 be the general solution of equation 1.9. Assume that (i) at least at one of the tails of $f(x)$ defined in the interval $0 \le x \le N$ vanishes and that its first derivative also vanishes, and (ii) the second solution is continuous at $x = R$ and $x = S$. Then $C_1 = C_2 = C_3 = 0$.

Proof: Since $H_1(x)$ is continuous at $x = R$ and $H_2(x)$ is continuous at $x = S$, then we have

$$H_1(R) = H_2(R) \quad \text{and} \quad H_2(S) = H_3(S),$$

or

$$C_1 h_1(R) = C_2 h_2(R), \tag{1.32}$$

and

$$C_2 h_2(S) = C_3 h_3(S). \tag{1.33}$$

Consider first the left tail of $f(x)$ to satisfy the assumptions, namely, $f(x)$ defined at $x = 0$ vanishes, i.e., $f_1(0) = 0$ and its first derivative also vanishes, i.e., $f'_1(0) = 0$. Then from equation 1.17 and Lemma 1.1, we have $C_1 = 0$. It immediately follows from equations 1.32 and 1.33 that

$$C_2 = 0 \quad \text{and} \quad C_3 = 0.$$

Hence $C_1 = C_2 = C_3 = 0$.

Similarly, consider second the right tail of $f(x)$ to satisfy the assumptions, namely, $f(x)$ defined at $x = N$ vanishes, i.e., $f_3(N) = 0$ and its first derivative also vanishes, i.e., $f'_3(N) = 0$. Then from equation 1.22 and Lemma 1.1, we have $C_3 = 0$. From equations 1.32 and 1.33, we obtain

$$C_1 = C_2 = C_3 = 0.$$

Likewise, consider third the tails of $f(x)$ to satisfy the assumptions, namely, $f(x)$ defined at $x = 0$ and $x = N$ vanish, i.e., $f_1(0) = 0$ and $f_3(N) = 0$ and its first derivative also vanish, i.e., $f'_1(0) = 0$ and $f'_3(N) = 0$. Then from equations 1.17 and 1.22 and Lemma 1.1, we have $C_1 = 0$ and $C_3 = 0$. Thus from equations 1.32 and 1.33, we obtain

$$C_1 = C_2 = C_3 = 0. \blacksquare$$

If the conditions of Theorem 1.1 are satisfied, then the general solution specified by equation 1.14 can be reduced to

$$f_i(x) = K_i \frac{\Omega_i(x)}{A_{2i}(x)}, \quad \text{for } i = 1, 2, 3, \tag{1.34}$$

where K_i are positive constants and $\Omega_i(x)$ is given in equation 1.15. Furthermore, the right hand side of equations 1.17 through 1.22 can be set to zero, namely,

$$- A_{11}(0)f_1(0) + \frac{1}{2}\frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=0} = 0, \tag{1.35}$$

$$- A_{11}(R)f_1(R) + \frac{1}{2}\frac{d}{dx}[A_{21}(x)f_1(x)]\Big|_{x=R} = 0, \tag{1.36}$$

$$- A_{12}(R)f_2(R) + \frac{1}{2}\frac{d}{dx}[A_{22}(x)f_2(x)]\Big|_{x=R} = 0, \tag{1.37}$$

$$- A_{12}(S)f_2(S) + \frac{1}{2}\frac{d}{dx}[A_{22}(x)f_2(x)]\Big|_{x=S} = 0, \tag{1.38}$$

$$- A_{13}(S)f_3(S) + \frac{1}{2}\frac{d}{dx}[A_{23}(x)f_3(x)]\Big|_{x=S} = 0, \tag{1.39}$$

$$- A_{13}(N)f_3(N) + \frac{1}{2}\frac{d}{dx}[A_{23}(x)f_3(x)]\Big|_{x=N} = 0. \tag{1.40}$$

We will show that the solution specified by equation 1.34 satisfies the differential equation 1.9 and equations 1.35 through 1.40.

Lemma 1.2

The solution

$$(1.41) \qquad f_i(x) = \frac{K_i}{A_{2i}(x)} \exp\left\{ \int^x \frac{2A_{1i}(y)}{A_{2i}(y)} dy \right\}, \; K_i > 0 \text{ and } 0 \le x \le N$$

of the differential equations 1.9, namely,

$$(1.42) \qquad \frac{d}{dx}\,[A_{1i}(x)f_i(x)] + \frac{1}{2}\,\frac{d^2}{dx^2}\,[A_{2i}(x)f_i(x)] = 0, \quad i = 1, 2, 3$$

is verified to satisfy equations 1.35 through 1.40.

proof: Multiplying on both sides of equation 1.41 by $A_{2i}(x)$, we obtain

$$(1.43) \qquad A_{2i}(x)f_i(x) = K_i \exp\left\{ \int^x \frac{2A_{1i}(y)}{A_{2i}(y)} dy \right\}, \quad \text{for } i = 1, 2, 3.$$

Taking derivative on both sides of equation 1.43, we obtain

$$(1.44) \qquad \frac{d}{dx}\,[A_{2i}(x)f_i(x)] = K_i \exp\left\{ \int^x \frac{2A_{1i}(y)}{A_{2i}(y)} dy \right\} \frac{2A_{1i}(x)}{A_{2i}(x)}$$

$$= 2A_{1i}(x)f_i(x), \quad \text{for } i = 1, 2, 3.$$

From equation 1.44, it is easy to show that equation 1.41 satisfies differential equation 1.42 for i = 1, 2, 3.

Next, we show that equation 1.41 satisfies equations 1.35 through 1.40. Substituting x = 0 and x = R into equation 1.44 for i = 1, respectively, we obtain

$$(1.45) \qquad \frac{d}{dx}\,[A_{21}(x)f_1(x)]\Big|_{x=0} = 2A_{11}(0)f_1(0),$$

and

$$(1.46) \qquad \frac{d}{dx}\,[A_{21}(x)f_1(x)]\Big|_{x=R} = 2A_{11}(R)f_1(R).$$

Therefore, from equations 1.45 and 1.46, equations 1.35 and 1.36

hold. Similarly, we can show that equation 1.41 for $i = 2$ and $i = 3$ satisfies equations 1.37 through 1.40. ∎

It should be noted that Lemma 1.1, Lemma 1.2 and theorem 1.1 also hold for the other case when $R > S$.

1.5 Literature Review

In this section we discuss the literature which is most relevant to the Machine Repair Problem (MRP) and Diffusion Approximation.

The M/M/R MRP has been the subject of investigation (i) for the no spares case by Saaty [51], Benson and Cox [6], Feller [15], and Page [48], (ii) for the cold standbys case by Taylor and Jackson [63], Toft and Boothroyd [66], Hilliard [30], Gross et al. [23], and (iii) for the warm standbys case by Albright [1, 2], and Sivazlian and Wang [57]. Exact steady-state solutions of the MRP are obtained for (i) the M/M/R model by Saaty [51], Benson and Cox [6], Feller [15], Hilliard [30], Gross and Harris [22], and Sivazlian and Wang [57], (ii) the M/G/1 and M/D/1 models by Ashcroft [4] and Takacs [62], (iii) the $M/E_k/1$ model by Benson and Cox [6], (iv) the D/D/R model by Allen [3], (v) the $M/E_K/R$ model by Maritas and Xirokostas [44], and (vi) the G/M/R model by Bunday and Scraton [9], and Sztrik [60, 61]. In fact, Bunday and Scraton [9], and Sztrik [60] show that the G/M/R results are the same as the M/M/R results. The approximate steady-state solutions for the M/D/R and M/G/R models by Tijms and Van Hoorn [65] and for the G/G/R model by Sivazlian and Wang [55]. Naor [46] uses normal approximation to the M/M/R MRP for large values of R and of N the number of machines.

Exact and tractable steady-state solutions for the more general G/G/R model are unknown. Recently, however, several authors have used diffusion approximation to investigate the G/G/R system. This includes Iglehart [31], Halachmi and Franta [25], Sunaga et al. [58], Haryono and Sivazlian [27], and Sivazlian and Wang [55]. The use of diffusion approximation to study the G/G/R MRP is due to Iglehart [31], Haryono and Sivazlian [27], and Sivazlian and Wang [55]. The main differences between the G/G/R system and the G/G/R MRP in using this type of approximation are (i) in the nature of the boundary conditions, and (ii) in the structure of the diffusion equations. For the multi-server queue, the G/G/R system has only one reflecting barrier, and two diffusion equations; on the other hand, the G/G/R MRP with spares has two reflecting barriers and at least three diffusion equations.

The use of the diffusion approximation in queueing systems has been studied by Cox and Miller [11], Gaver and Shedler [18], Kobayashi [41, 42], Gelenbe [19], Kleinrock [40], Halfin and Whitt [26], Karlin and Taylor [32], Karmeshu and Jaiswal [33], Falin [13], Newell [47], and Gelenbe and Pujolle [20]. The diffusion approximation is based on the assumption that queues of failed machines in the repair stage are almost always non-empty. The first paper applying the heavy traffic approximations to the many server queueing systems is due to Kingman [38]. Whitt [71] provides many references on queue in heavy traffic. The diffusion approximation to the G/G/1 systems under heavy traffic condition is given in Heyman [28]. Whitt [72] investigates the mean waiting time for the G/G/1 queue using different diffusion approximations, yielding different results. Halachmi and Franta [25] develop a diffusion approximation model for the G/G/R queue which is

consistent with some known heavy traffic limit theorems. Kimura [35] proposes a diffusion approximation model for the M/G/R queue; the numerical results obtained indicate that his model is more accurate than the earlier diffusion model which is derived by Halachmi and Franta [25]. Recently, Yao [73] refines Kimura's diffusion model for the M/G/R queue and his results show a significant improvement of Kimura's model. Under heavy traffic conditions, Halachmi and Franta [25] and Sunaga et al. [58] investigate the mean of the number of customers in the G/G/R system.

Diffusion models can yield simple explicit but approximate solution to certain models in which boundary conditions are imposed. Reflecting barrier(s) or some other appropriate boundary conditions should be imposed on the diffusion equations to obtain the required solution. In the models for growth distributions of biological organisms, Saaty [52] investigates a special case of birth and death process with two absorbing barriers. Cox and Miller [11], and Sweet and Hardin [59], propose approximate solutions of the Fokker-Planck equations for some diffusion processes when two reflecting barriers or two absorbing barriers or one reflecting and one absorbing barrier are present. Kobayashi [42] considers two reflecting barriers for the cyclic queueing systems.

Cost (Profit) models of the M/M/R MRP have been studied (i) for the no spare case by Ashcroft [4], Morse [45], White et al. [70], and Elsayed [12], (ii) for the cold-standby case by Hilliard [30], Gross et al. [23, 24], and (iii) for the warm standbys case by Albright [1, 2], Sivazlian and Wang [57], and Wang and Sivazlian [69]. Moreover, profit

model in the G/G/R machine repair model is investigated by Sivazlian and Wang [56].

The problem of calculating the machine availability and the operative utilization in the situation where there are only a group of identical operating machines (no spares) with R repairmen, has been treated by (i) Benson and Cox [6] for the M/M/R model, and (ii) Maritas and Xirokostas [44] for the $M/E_k/R$ model. Using the probability density function f(x) of the G/G/R MRP with warm-standby spares through the theory of diffusion approximation to calculate the machine availability and the operative utilization for the G/G/R model.

1.6 Scope of the Study

In Chapter 2, we study the M/M/R warm-standby machine repair problem (MRP) under steady-state conditions. We develop the differential difference equations for the M/M/R MRP and then derive the exact steady state solutions P_n, the probability of n failed machines in the system. Next, we formulate the diffusion equations approximating P_n, when n is treated as a continuous variable. Finally, we propose a heuristic approximation to obtain the infinitesimal mean and variance (or diffusion parameters) which are used to set up the diffusion equations for the probability density functions for the queue length in the M/M/R model.

In Chapter 3, we propose an approximation to the G/G/R MRP with warm standbys. The approximate diffusion parameters of the diffusion equations for the G/G/R MRP with warm standbys are obtained under heavy traffic approximations using well-known asymptotic results from renewal theory. Diffusion parameters are used to set up the steady-state

diffusion equations. These equations are solved analytically to obtain approximate formulas for the probability density functions for the queue length in the G/G/R MRP with warm standbys. Several figures for the approximate probability density function for a range of the problem parameters are plotted, and their characteristics are analyzed for the M/M/R, M/G/R, G/M/R, and G/G/R machine repair models with warm standbys.

In Chapter 4, we use the results from Chapter 2 and Chapter 3 to compute the system characteristics for the M/M/R and G/G/R machine repair models with warm standbys, respectively. A profit model is developed to determine the optimal values of the number of repairmen and the number of spares simultaneously to maximize the steady-state expected profit per unit time. Results are presented in which system characteristics are calculated under optimal operating conditions.

In Chapter 5, a comparison for the system characteristics MRP with no spares is made between the results obtained from the diffusion approximation, and some of the existing exact results which have appeared in the literature, including expected number of failed machines, machine availability, operative utilization, etc. The diffusion approximation approach is accurate enough for practical purposes and is a useful method for solving complex machine repair problems.

In Chapter 6, a comparison for the system characteristics is made between the results obtained from diffusion approximation, from simulation results, and from analytic results for the M/M/R and the $E_2/E_2/R$ machine repair problems with warm standbys. Different selection modes are considered for choosing spares to replace failed operating machines. The simulation results show that the particular selection

mode is not a significant factor. It is also shown that the cut-off time when analyzing steady state condition should be considered as a significant factor in simulation. We provide a sensitivity analysis for the system characteristics to the G/G/R MRP with warm standbys based on varying the square coefficients of variation of the succession of the uptimes of the operating machines, the uptimes of the spare machines, and the repair times. The results show that the model for the M/M/R MRP provides an excellent approximation to obtain the system characteristics for the G/G/R MRP.

In Chapter 7, we consider an application of the warm-standby machine repair problem with slow and fast repair rates. The impact of the number of spares for various levels of the repair facility on system availability is studied. Finally, we propose a number of future research areas related to the present work.

CHAPTER 2

THE M/M/R MACHINE REPAIR PROBLEM WITH WARM STANDBYS

The M/M/R machine repair problem (MRP) with cold standbys was first considered by Taylor and Jackson [63]. In this chapter, we extend the previous work to the analysis of the MRP with warm standbys. We first develop the exact warm-standby model for the M/M/R MRP and show that it generalizes the existing no spares and cold standbys models. The results are more general than existing results and may be used to obtain the results for the no spares or cold standbys as a special case. Next, we use the idea of the diffusion approximation to derive the probability density function of the diffusion model for the M/M/R MRP. Finally, we develop a heuristic derivation for the infinitesimal mean and variance for the M/M/R queue length over small time interval. This may suggest that an appropriate choice of infinitesimal mean and variance will provide a good approximation to the queue length of the G/G/R MRP.

2.1 Development of the Equations and Solutions

Let n represent the number of failed machines in the system. The mean arrival rate λ_n and the mean repair rate μ_n for the M/M/R MRP with warm standbys are given by

$$\lambda_n = \begin{cases} M\lambda + (S - n)\alpha, & \text{for } n = 0, 1, 2, \ldots, S, \\ (N - n)\lambda, & \text{for } n = S+1, S+2, \ldots, M+S-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & \text{for } n = 1, 2, \ldots, R-1, \\ R\mu, & \text{for } n = R, R+1, \ldots, M+S=N, \\ 0, & \text{otherwise.} \end{cases}$$

2.1.1 Mathematical Development of the Differential Difference Equations for the M/M/R Model

Let $\{N(t), t \geq 0\}$ denote the number of failed machines in the system (waiting and in repair) at time t. Let

$$P(n;t) = \text{prob}\{N(t) = n\}, \quad \text{for } n = 0, 1, 2, \ldots, N,$$

be the probability of exactly n failed machines in the system at time t. P(n;t) satisfies the following system of equations for sufficiently small dt,

(i) when $R \leq S$

$$\begin{aligned} P(0;t+dt) = {} & P(0;t)\,[1 - (M\lambda + S\alpha)dt] \\ & + P(1;t)\,\{1 - [M\lambda + (S-1)\alpha]dt\}\,\mu dt \\ & + o(dt), \end{aligned}$$

$$\begin{aligned} P(n;t+dt) = {} & P(n;t)\,\{1 - [M\lambda + (S-n)\alpha]dt\}\,(1 - n\mu dt) \\ & + P(n+1;t)\,\{1 - [M\lambda + (S-n-1)\alpha]dt\}\,(n+1)\mu dt \\ & + P(n-1;t)\,\{[M\lambda + (S-n+1)\alpha]dt\}\,[1 - (n-1)\mu dt] \\ & + o(dt), \qquad 1 \leq n < R, \end{aligned}$$

$$\begin{aligned} P(n;t+dt) = {} & P(n;t)\,\{1 - [M\lambda + (S-n)\alpha]dt\}\,(1 - R\mu dt) \\ & + P(n+1;t)\,\{1 - [M\lambda + (S-n-1)\alpha]dt\}\,R\mu dt \\ & + P(n-1;t)\,\{[M\lambda + (S-n+1)\alpha]dt\}\,(1 - R\mu dt) \\ & + o(dt), \qquad R \leq n \leq S, \end{aligned}$$

$$P(n;t+dt) = P(n;t)\ [1 - (N-n)\lambda dt]\ (1 - R\mu dt)$$
$$+ P(n+1;t)\ [1 - (N-n-1)\lambda dt]\ R\mu dt$$
$$+ P(n-1;t)\ [(N-n+1)\lambda dt]\ (1 - R\mu dt)$$
$$+ o(dt), \qquad S < n < N,$$

$$P(n;t+dt) = P(n;t)\ (1 - R\mu dt)$$
$$+ P(n-1;t)\ \lambda dt\ (1 - R\mu dt)$$
$$+ o(dt), \qquad n = N,$$

and (ii) when $R > S$

$$P(0;t+dt) = P(0;t)\ [1 - (M\lambda + S\alpha)dt]$$
$$+ P(1;t)\ \{1 - [M\lambda + (S-1)\alpha]dt\}\ \mu dt$$
$$+ o(dt),$$

$$P(n;t+dt) = P(n;t)\ \{1 - [M\lambda + (S-n)\alpha]dt\}\ (1 - n\mu dt)$$
$$+ P(n+1;t)\ \{1 - [M\lambda + (S-n-1)\alpha]dt\}\ (n+1)\mu dt$$
$$+ P(n-1;t)\ \{[M\lambda + (S-n+1)\alpha]dt\}\ [1 - (n-1)\mu dt]$$
$$+ o(dt), \qquad 1 \le n \le S,$$

$$P(n;t+dt) = P(n;t)\ [1 - (N-n)\lambda]dt\ (1 - n\mu dt)$$
$$+ P(n+1;t)\ [1 - (N-n-1)\lambda]dt\ (n+1)\mu dt$$
$$+ P(n-1;t)\ (N-n+1)\lambda dt\ [1 - (n-1)\mu dt)$$
$$+ o(dt), \qquad S < n < R,$$

$$P(n;t+dt) = P(n;t)\ [1 - (N-n)\lambda dt]\ (1 - R\mu dt)$$
$$+ P(n+1;t)\ [1 - (N-n-1)\lambda dt]\ R\mu dt$$
$$+ P(n-1;t)\ [(N-n+1)\lambda dt]\ (1 - R\mu dt)$$
$$+ o(dt), \qquad R \le n < N,$$

$$P(n;t+dt) = P(n;t)\ (1 - R\mu dt)$$
$$+ P(n-1;t)\ \lambda dt\ (1 - R\mu dt)$$
$$+ o(dt), \qquad n = N.$$

Putting

$$\frac{dP(n;t)}{dt} = \lim_{dt \to 0} \frac{P(n;t+dt) - P(n;t)}{dt},$$

and using the fact that

$$\lim_{dt \to 0} \frac{o(dt)}{dt} = 0,$$

we obtain the following differential difference equations

(i) when $R \leq S$

(2.1a) $$\frac{dP(0;t)}{dt} = -(M\lambda + S\alpha)\, P(0;t) + \mu\, P(1;t),$$

(2.1b) $$\frac{dP(n;t)}{dt} = -[M\lambda + (S - n)\alpha + n\mu]\, P(n;t) + (n + 1)\mu\, P(n+1;t) + [M\lambda + (S - n + 1)\alpha]\, P(n-1;t), \quad 0 < n < R,$$

(2.1c) $$\frac{dP(n;t)}{dt} = -[M\lambda + (S - n)\alpha + R\mu]\, P(n;t) + R\mu\, P(n+1;t) + [M\lambda + (S - n + 1)\alpha]\, P(n-1;t), \quad R \leq n \leq S,$$

(2.1d) $$\frac{dP(n;t)}{dt} = -[(N - n)\lambda + R\mu]\, P(n;t) + R\mu\, P(n+1;t) + [(N - n + 1)\lambda]\, P(n-1;t), \quad S < n < N,$$

(2.1e) $$\frac{dP(n;t)}{dt} = -R\mu\, P(n;t) + \lambda\, P(n-1;t), \quad n = N,$$

and (ii) when $R > S$

(2.2a) $$\frac{dP(0;t)}{dt} = -(M\lambda + S\alpha)\, P(0;t) + \mu\, P(1;t),$$

(2.2b) $$\frac{dP(n;t)}{dt} = -[M\lambda + (S - n)\alpha + n\mu]\, P(n;t) + (n + 1)\mu\, P(n+1;t) + [M\lambda + (S - n + 1)\alpha]\, P(n-1;t), \quad 0 < n \leq S,$$

$$(2.2c)\qquad \frac{dP(n;t)}{dt} = -\left[(N-n)\lambda + n\mu\right] P(n;t) + (n+1)\mu\, P(n+1;t)$$
$$+ \left[(N-n+1)\lambda\right] P(n-1;t), \quad S < n < R,$$

$$(2.2d)\qquad \frac{dP(n;t)}{dt} = -\left[(N-n)\lambda + R\mu\right] P(n;t) + R\mu\, P(n+1;t)$$
$$+ \left[(N-n+1)\lambda\right] P(n-1;t), \quad R \le n < N,$$

$$(2.2e)\qquad \frac{dP(n;t)}{dt} = -R\mu\, P(n;t) + \lambda\, P(n-1;t), \quad n = N.$$

There is another way to derive the differential difference equations 2.1 and 2.2 above by using the following state-transition rate diagram (sometimes called a "balance diagram") such as Figure 2.1 when $R \le S$ and Figure 2.2 when $R > S$, respectively.

2.1.2 The Exact Model and Solutions for the M/M/R Model in Steady-State

The solution for the steady-state, if it exists, must satisfy

$$\lim_{t \to \infty} \frac{dP(n;t)}{dt} = 0,$$

i.e., $P(n;t)$ is independent of t, and if we let

$$P_n = \lim_{t \to \infty} P(n;t),$$

where

P_0 = probability that no machines are broken down, and

P_n = probability that there are n failed machines in the system, where $n = 1, \ldots, N$.

The equilibrium equations for P_n are given by

(i) when $R \le S$

(2.3a) $$(M\lambda+S\alpha)\ P_0 = \mu\ P_1,$$

(2.3b) $$[M\lambda+(S-n)\alpha+n\mu]\ P_n = [M\lambda+(S-n+1)\alpha]\ P_{n-1} + (n+1)\mu\ P_{n+1}, \quad 1 \le n < R,$$

(2.3c) $$[M\lambda+(S-n)\alpha+R\mu]\ P_n = [M\lambda+(S-n+1)\alpha]\ P_{n-1} + R\mu\ P_{n+1}, \quad R \le n \le S,$$

(2.3d) $$[(N-n)\lambda+R\mu]\ P_n = [(N-n+1)\lambda]\ P_{n-1} + R\mu\ P_{n+1}, \quad S < n < N,$$

(2.3e) $$\lambda\ P_{N-1} = R\mu\ P_N,$$

and (ii) when $R > S$

(2.4a) $$(M\lambda+S\alpha)\ P_0 = \mu\ P_1$$

(2.4b) $$[M\lambda+(S-n)\alpha+n\mu]\ P_n = [M\lambda+(S-n+1)\alpha]\ P_{n-1} + (n+1)\mu\ P_{n+1}, \quad 1 \le n \le S,$$

(2.4c) $$[(N-n)\lambda+n\mu]\ P_n = [(N-n+1)\lambda]\ P_{n-1} + (n+1)\mu\ P_{n+1}, \quad S < n < R,$$

(2.4d) $$[(N-n)\lambda+R\mu]\ P_n = [(N-n+1)\lambda]\ P_{n-1} + R\mu\ P_{n+1}, \quad R \le n < N,$$

(2.4e) $$\lambda\ P_{N-1} = R\mu\ P_N.$$

Using the general birth and death results given by equation 2.5

(2.5) $$P_n = \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j}\ P_0,$$

or solving equations 2.3 and 2.4 recursively, we obtain the steady state solutions respectively

(i) when $R \le S$

(2.6a) $$P_n = \frac{1}{n!} \prod_{j=0}^{n-1} [M\theta_\lambda + (S - j)\theta_\alpha]\ P_0, \quad 1 \le n < R,$$

(2.6b) $$P_n = \frac{1}{R!\ R^{n-R}} \prod_{j=0}^{n-1} [M\theta_\lambda + (S - j)\theta_\alpha]\ P_0, \quad R \le n \le S,$$

(2.6c) $$P_n = \frac{(M-1)!\ \theta_\lambda^{n-S-1}}{(N-n)!\ R!\ R^{n-R}} \prod_{j=0}^{S} [M\theta_\lambda + (S - j)\theta_\alpha]\ P_0, \quad S < n \le N,$$

and (ii) when R > S

$$(2.7a)\qquad P_n = \frac{1}{n!} \prod_{j=0}^{n-1} [M\theta_\lambda + (S - j)\theta_\alpha] P_0, \qquad 1 \le n \le S,$$

$$(2.7b)\qquad P_n = \frac{(M-1)!\ \theta_\lambda^{n-S-1}}{(N-n)!\ n!} \prod_{j=0}^{S} [M\theta_\lambda + (S - j)\theta_\alpha] P_0, \qquad S < n < R,$$

$$(2.7c)\qquad P_n = \frac{(M-1)!\ \theta_\lambda^{n-S-1}}{(N-n)!\ R!\ R^{n-R}} \prod_{j=0}^{S} [M\theta_\lambda + (S - j)\theta_\alpha] P_0, \qquad R \le n \le N,$$

where

$$\theta_\lambda = \frac{\lambda}{\mu} \quad \text{and} \quad \theta_\alpha = \frac{\alpha}{\mu}.$$

It should be noted that i) when $\alpha = 0$, equations 2.6 and 2.7 for P_n reduce to the existing results for the cold standbys model in the literature (see, Toft and Boothroyd [66]), ii) when $S = 0$, and $\alpha = 0$, equations 2.7b and 2.7c for P_n reduce to the existing results for the no spares model in the literature (see, Feller [15]).

The steady-state solutions P_n always exist in the system because the number of states is finite. For both equations 2.6 and 2.7, P_0 can be solved from the normalizing equation

$$\sum_{n=0}^{N} P_n = 1.$$

Therefore, equations 2.6 and 2.7 are used to obtain the steady-state solutions P_n for the M/M/R MRP with warm standbys shown in Table 2.1 when $R \le S$ and in Table 2.2 when $R > S$, for the range $0.2 \le \theta_\lambda \le 1.4$ in increments of 0.2 and for various number of repairmen R.

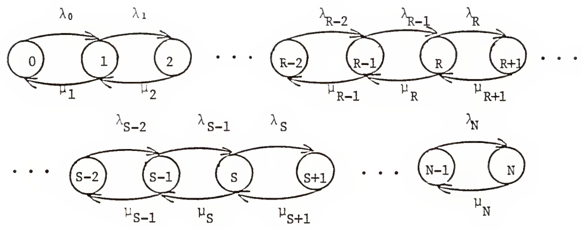


Figure 2.1 State-transition rate diagram for the M/M/R machine repair problem with warm standby spares.

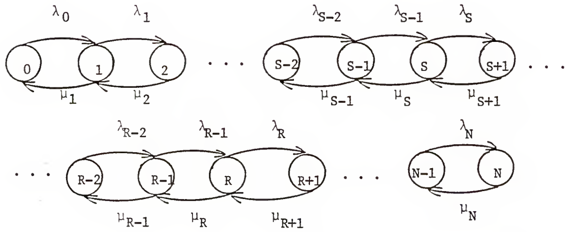


Figure 2.2 State-transition rate diagram for the M/M/R machine repair problem with warm standby spares.

Table 2.1 Steady-state solutions P_n of the exact model for the M/M/R MRP with warm standbys (M=10, S=5, N=15, θ_α=0.05).

	θ_λ						(R = 3)
n	0.2	0.4	0.6	0.8	1.0	1.2	1.4
0	0.0925	0.0037	0.0001	0.0000	0.0000	0.0000	0.0000
1	0.2082	0.0156	0.0007	0.0000	0.0000	0.0000	0.0000
2	0.2290	0.0328	0.0022	0.0002	0.0000	0.0000	0.0000
3	0.1641	0.0454	0.0044	0.0005	0.0001	0.0000	0.0000
4	0.1149	0.0620	0.0090	0.0013	0.0002	0.0001	0.0000
5	0.0785	0.0837	0.0181	0.0036	0.0008	0.0002	0.0001
6	0.0523	0.1116	0.0362	0.0095	0.0027	0.0009	0.0003
7	0.0314	0.1339	0.0651	0.0228	0.0081	0.0031	0.0013
8	0.0167	0.1428	0.1042	0.0486	0.0216	0.0099	0.0048
9	0.0078	0.1333	0.1458	0.0908	0.0504	0.0278	0.0158
10	0.0031	0.1066	0.1750	0.1453	0.1008	0.0668	0.0442
11	0.0010	0.0711	0.1750	0.1937	0.1680	0.1336	0.1031
12	0.0003	0.0379	0.1400	0.2066	0.2240	0.2138	0.1924
13	0.0001	0.0152	0.0840	0.1653	0.2240	0.2565	0.2694
14	0.0000	0.0040	0.0336	0.0882	0.1494	0.2052	0.2514
15	0.0000	0.0005	0.0067	0.0235	0.0498	0.0821	0.1173

Table 2.1--Continued

	θ_λ						(R = 5)
n	0.2	0.4	0.6	0.8	1.0	1.2	1.4
0	0.1106	0.0141	0.0015	0.0002	0.0000	0.0000	0.0000
1	0.2488	0.0598	0.0094	0.0013	0.0002	0.0000	0.0000
2	0.2737	0.1257	0.0293	0.0055	0.0011	0.0002	0.0001
3	0.1961	0.1739	0.0600	0.0149	0.0036	0.0009	0.0003
4	0.1030	0.1782	0.0915	0.0301	0.0090	0.0028	0.0009
5	0.0422	0.1443	0.1107	0.0485	0.0181	0.0067	0.0026
6	0.0169	0.1155	0.1328	0.0776	0.0363	0.0162	0.0073
7	0.0061	0.0831	0.1434	0.1117	0.0653	0.0349	0.0185
8	0.0019	0.0532	0.1377	0.1430	0.1044	0.0671	0.0413
9	0.0005	0.0298	0.1157	0.1601	0.1462	0.1127	0.0810
10	0.0001	0.0143	0.0833	0.1537	0.1754	0.1622	0.1361
11	0.0000	0.0057	0.0500	0.1230	0.1754	0.1947	0.1906
12	0.0000	0.0018	0.0240	0.0787	0.1403	0.1869	0.2134
13	0.0000	0.0004	0.0086	0.0378	0.0842	0.1346	0.1793
14	0.0000	0.0001	0.0021	0.0121	0.0337	0.0646	0.1004
15	0.0000	0.0000	0.0002	0.0019	0.0067	0.0155	0.0281

Table 2.2 Steady-state solutions P_n of the exact model for the M/M/R MRP with warm standbys (M=10, S=5, N=15, θ_α=0.05).

	θ_λ						(R = 6)
n	0.2	0.4	0.6	0.8	1.0	1.2	1.4
0	0.1112	0.0156	0.0021	0.0003	0.0000	0.0000	0.0000
1	0.2503	0.0662	0.0133	0.0025	0.0005	0.0001	0.0000
2	0.2753	0.1391	0.0414	0.0102	0.0024	0.0006	0.0002
3	0.1973	0.1924	0.0848	0.0276	0.0082	0.0025	0.0008
4	0.1036	0.1973	0.1293	0.0559	0.0208	0.0075	0.0028
5	0.0425	0.1598	0.1564	0.0899	0.0418	0.0182	0.0079
6	0.0142	0.1065	0.1564	0.1199	0.0696	0.0364	0.0185
7	0.0042	0.0639	0.1408	0.1439	0.1044	0.0655	0.0389
8	0.0011	0.0341	0.1166	0.1535	0.1392	0.1048	0.0726
9	0.0003	0.0159	0.0789	0.1433	0.1624	0.1467	0.1186
10	0.0001	0.0064	0.0473	0.1146	0.1624	0.1760	0.1660
11	0.0000	0.0021	0.0237	0.0764	0.1353	0.1760	0.1936
12	0.0000	0.0006	0.0095	0.0408	0.0902	0.1408	0.1807
13	0.0000	0.0001	0.0028	0.0163	0.0451	0.0845	0.1265
14	0.0000	0.0000	0.0006	0.0043	0.0150	0.0338	0.0590
15	0.0000	0.0000	0.0001	0.0006	0.0025	0.0068	0.0138

Table 2.2--Continued

	θ_λ						(R = 8)
n	0.2	0.4	0.6	0.8	1.0	1.2	1.4
0	0.1114	0.0162	0.0025	0.0004	0.0001	0.0000	0.0000
1	0.2506	0.0687	0.0159	0.0037	0.0009	0.0002	0.0001
2	0.2756	0.1443	0.0492	0.0151	0.0046	0.0015	0.0005
3	0.1975	0.1996	0.1008	0.0411	0.0157	0.0060	0.0023
4	0.1037	0.2046	0.1538	0.0833	0.0397	0.0181	0.0082
5	0.0425	0.1657	0.1861	0.1341	0.0798	0.0436	0.0231
6	0.0142	0.1105	0.1861	0.1788	0.1330	0.0873	0.0538
7	0.0036	0.0568	0.1435	0.1839	0.1710	0.1346	0.0969
8	0.0007	0.0227	0.0861	0.1471	0.1710	0.1616	0.1357
9	0.0001	0.0080	0.0452	0.1030	0.1497	0.1696	0.1662
10	0.0000	0.0024	0.0203	0.0618	0.1122	0.1527	0.1745
11	0.0000	0.0006	0.0076	0.0309	0.0702	0.1145	0.1527
12	0.0000	0.0001	0.0023	0.0124	0.0351	0.0687	0.1069
13	0.0000	0.0000	0.0005	0.0037	0.0132	0.0309	0.0561
14	0.0000	0.0000	0.0001	0.0007	0.0033	0.0093	0.0196
15	0.0000	0.0000	0.0000	0.0001	0.0004	0.0014	0.0034

Let

$$E[N] \equiv \sum_{n=0}^{N} nP_n$$

represent the expected number of failed machines in the M/M/R with warm standbys in steady-state. Results for $E[N]$ are listed in Table 2.3 when $R \le S$ and in Table 2.4 when $R > S$ for a range of the problem parameters and for various values of R.

Table 2.3 The expected number of failed machines for the exact model for the M/M/R MRP with warm standbys in steady-state (M=10, S=5, N=15).

θ_α=0.05	R				
θ_λ	1	2	3	4	5
0.2	9.9805	4.9631	2.7956	2.2921	2.1735
0.4	12.4999	9.9782	7.3972	5.4248	4.4707
0.6	13.3333	11.6656	9.9812	8.3067	6.9250
0.8	13.7500	12.4999	11.2475	9.9883	8.7758
1.0	14.0000	13.0000	11.9996	10.9970	10.0010
1.2	14.1667	13.3333	12.4999	11.6659	10.8326
1.4	14.2857	13.5714	12.8571	12.1426	11.4282
1.6	14.3750	13.7500	13.1250	12.4999	11.8748
1.8	14.4444	13.8889	13.3333	12.7778	12.2221
2.0	14.5000	14.0000	13.5000	13.0000	12.5000

Table 2.3--Continued

θ_α=0.1	R				
θ_λ	1	2	3	4	5
0.2	9.9885	5.1594	2.9824	2.4399	2.3085
0.4	12.4999	9.9825	7.4349	5.4906	4.5375
0.6	13.3333	11.6657	9.9838	8.3182	6.9453
0.8	13.7500	12.4999	11.2477	9.9899	8.7803
1.0	14.0000	13.0000	11.9996	10.9973	10.0020
1.2	14.1667	13.3333	12.4999	11.6659	10.8329
1.4	14.2857	13.5714	12.8571	12.1426	11.4282
1.6	14.3750	13.7500	13.1250	12.4999	11.8748
1.8	14.4444	13.8889	13.3333	12.7778	12.2221
2.0	14.5000	14.0000	13.5000	13.0000	12.5000

Table 2.4 The expected number of failed machines for the exact model for the M/M/R MRP with warm standbys in steady-state (M=10, S=5, N=15).

θ_α=0.05	R				
θ_λ	6	7	8	9	10
0.2	2.1461	2.1405	2.1396	2.1394	2.1394
0.4	4.1074	3.9849	3.9485	3.9393	3.9374
0.6	6.0775	5.6680	5.5016	5.4439	5.4272
0.8	7.7807	7.1312	6.7895	6.6404	6.5863
1.0	9.0761	8.3408	7.8643	7.6125	7.5032
1.2	10.0223	9.3029	8.7602	8.4245	8.2553
1.4	10.7214	10.0568	9.5014	9.1121	8.8894
1.6	11.2525	10.6499	10.1119	9.6971	9.4332
1.8	11.6676	11.1225	10.6153	10.1955	9.9038
2.0	12.0004	11.5055	11.0328	10.6209	10.3135

Table 2.4--Continued

θ_α=0.1	R				
θ_λ	6	7	8	9	10
0.2	2.2778	2.2715	2.2704	2.2702	2.2702
0.4	4.1707	4.0464	4.0093	3.9999	3.9980
0.6	6.1009	5.6916	5.5250	5.4672	5.4505
0.8	7.7877	7.1394	6.7980	6.6489	6.5947
1.0	9.0780	8.3435	7.8673	7.6156	7.5064
1.2	10.0228	9.3038	8.7613	8.4257	8.2565
1.4	10.7216	10.0571	9.5018	9.1126	8.8899
1.6	11.2526	10.6500	10.1121	9.6973	9.4334
1.8	11.6676	11.1226	10.6154	10.1955	9.9039
2.0	12.0004	11.5055	11.0329	10.6209	10.3135

2.1.3 The Diffusion Model and Solutions for the M/M/R Model in Steady-State

The idea of the diffusion approximation is to replace equations 2.1 and equations 2.2 by a system of partial differential equations that are easier to solve. We do this by replacing the discrete random variable n by the continuous random variable x and P(n;t) by f(x;t) in equations 2.1 and equations 2.2. Therefore, equations 2.1 become

(2.8a) $$\frac{\partial f(0;t)}{\partial t} = - [M\lambda + S\alpha]\ f(0;t) + \mu\ f(1;t),$$

(2.8b) $$\frac{\partial f(x;t)}{\partial t} = - [M\lambda + (S - x)\alpha + x\mu]\ f(x;t) + (x + 1)\mu\ f(x+1;t) + [M\lambda + (S - x + 1)\alpha]\ f(x-1;t), \quad 0 < x < R,$$

(2.8c) $$\frac{\partial f(x;t)}{\partial t} = - [M\lambda + (S - x)\alpha + R\mu]\ f(x;t) + R\mu\ f(x+1;t) + [M\lambda + (S - x + 1)\alpha]\ f(x-1;t), \quad R \le x \le S,$$

(2.8d) $$\frac{\partial f(x;t)}{\partial t} = - [(N - x)\lambda + R\mu]\ f(x;t) + R\mu\ f(x+1;t) + [(N - x + 1)\lambda]\ f(x-1;t), \quad S < x < N,$$

(2.8e) $$\frac{\partial f(x;t)}{\partial t} = - R\mu\ f(x;t) + \lambda\ f(x-1;t), \quad x = N,$$

and equations 2.2 become

(2.9a) $$\frac{\partial f(0;t)}{\partial t} = - (M\lambda + S\alpha)\ f(0;t) + \mu\ f(1;t)$$

(2.9b) $$\frac{\partial f(x;t)}{\partial t} = - [M\lambda + (S - x)\alpha + x\mu]\ f(x;t) + (x + 1)\mu\ f(x+1;t) + [M\lambda + (S - x + 1)\alpha]\ f(x-1;t), \quad 0 < x \le S,$$

(2.9c) $$\frac{\partial f(x;t)}{\partial t} = - [(N - x)\lambda + x\mu]\ f(x;t) + (x + 1)\mu\ f(x+1;t) + [(N - x + 1)\lambda]\ f(x-1;t), \quad S < x < R,$$

(2.9d) $$\frac{\partial f(x;t)}{\partial t} = - [(N - x)\lambda + R\mu]\ f(x;t) + R\mu\ f(x+1;t) + [(N - x + 1)\lambda]\ f(x-1;t), \quad R \le x < N,$$

(2.9e) $$\frac{\partial f(x;t)}{\partial t} = - R\mu\ f(x;t) + \lambda\ f(x-1;t), \quad x = N.$$

In Appendix A, we show that equation 2.8b reduces to equation 2.8a at $x = 0$, and that equation 2.8d reduces to equation 2.8e at $x = N$. Thus, equations 2.8b and 2.8d are sufficient to describe the process by setting $0 \le x < R$ and $S < x \le N$ for equations 2.8b and 2.8d, respectively. Similar proof in Appendix A shows that equation 2.9b reduces to equation 2.9a at $x = 0$ and equation 2.9d reduces to equation 2.9e at $x = N$. Thus, equations 2.9b and 2.9d are sufficient to describe the process by setting $0 \le x \le S$ and $R < x \le N$ for equations 2.9a and 2.9e, respectively. Let us expand the terms of the right hand side of equations 2.8 and 2.9 in a Taylor polynomial about x and retain only terms of the first- and second-order. We can obtain the following well-known diffusion equations

(i) when $R \le S$

$$(2.10a)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[M\lambda + (S-x)\alpha - x\mu]\, f(x;t)\} + \frac{\partial^2}{\partial x^2}\{\frac{M\lambda + (S-x+1)\alpha + (x+1)\mu}{2}\, f(x;t)\},\quad 0 \le x < R,$$

$$(2.10b)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[M\lambda + (S-x)\alpha - R\mu]\, f(x;t)\} + \frac{\partial^2}{\partial x^2}\{\frac{M\lambda + (S-x+1)\alpha + R\mu}{2}\, f(x;t)\},\quad R \le x \le S,$$

$$(2.10c)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[(N-x)\lambda - R\mu]\, f(x;t)\} + \frac{\partial^2}{\partial x^2}\{\frac{(N-x+1)\lambda + R\mu}{2}\, f(x;t)\},\quad S < x \le N.$$

and (ii) when R > S

$$(2.11a)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[M\lambda + (S-x)\alpha - x\mu]\ f(x;t)\}$$
$$+ \frac{\partial^2}{\partial x^2}\{\frac{M\lambda + (S-x+1)\alpha + (x+1)\mu}{2}\ f(x;t)\},\ 0 \le x \le S,$$

$$(2.11b)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[(N-x)\lambda - x\mu]\ f(x;t)\}$$
$$+ \frac{\partial^2}{\partial x^2}\{\frac{(N-x+1)\lambda + (x+1)\mu}{2}\ f(x;t)\},\quad S < x < R,$$

$$(2.11c)\quad \frac{\partial f(x;t)}{\partial t} = -\frac{\partial}{\partial x}\{[(N-x)\lambda - R\mu]\ f(x;t)\}$$
$$+ \frac{\partial^2}{\partial x^2}\{\frac{(N-x+1)\lambda + R\mu}{2}\ f(x;t)\},\quad R \le x \le N.$$

The probability density function f(x;t) has to be divided into three different ranges of x for the respective cases when $R \le S$ and when $R > S$, namely,

(i) when $R \le S$

$$(2.12)\qquad f(x;t) = \begin{cases} f_1(x;t), & \text{for } 0 \le x < R, \\ f_2(x;t), & \text{for } R \le x \le S, \\ f_3(x;t), & \text{for } S < x \le N, \end{cases}$$

and (ii) when $R > S$

$$(2.13)\qquad f(x;t) = \begin{cases} f_4(x;t), & \text{for } 0 \le x \le S, \\ f_5(x;t), & \text{for } S < x < R, \\ f_6(x;t), & \text{for } R \le x \le N. \end{cases}$$

Under steady-state conditions, we let

$$(2.14)\qquad f_i(x) = \lim_{t \to \infty} f_i(x;t),\quad \text{for } i = 1, 2, \ldots, 6.$$

Equations 2.10 and equations 2.11 become

(i) when $R \le S$

(2.15a) $$0 = -\frac{d}{dx}\{[M\lambda + (S-x)\alpha - x\mu]\ f_1(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda + (S-x+1)\alpha + (x+1)\mu}{2}\ f_1(x)\}, \quad 0 \le x < R,$$

(2.15b) $$0 = -\frac{d}{dx}\{[M\lambda + (S-x)\alpha - R\mu]\ f_2(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda + (S-x+1)\alpha + R\mu}{2}\ f_2(x)\}, \quad R \le x \le S,$$

(2.15c) $$0 = -\frac{d}{dx}\{[(N-x)\lambda - R\mu]\ f_3(x)\} + \frac{d^2}{dx^2}\{\frac{(N-x+1)\lambda + R\mu}{2}\ f_3(x)\}, \quad S < x \le N,$$

and (ii) when $R > S$

(2.16a) $$0 = -\frac{d}{dx}\{[M\lambda + (S-x)\alpha - x\mu]\ f_4(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda + (S-x+1)\alpha + (x+1)\mu}{2}\ f_4(x)\}, \quad 0 \le x \le S,$$

(2.16b) $$0 = -\frac{d}{dx}\{[(N-x)\lambda - x\mu]\ f_5(x)\} + \frac{d^2}{dx^2}\{\frac{(N-x+1)\lambda + (x+1)\mu}{2}\ f_5(x)\}, \quad S < x < R,$$

$$(2.16c) \qquad 0 = - \frac{d}{dx} \{[(N-x)\lambda - R\mu]\ f_6(x)\} + \frac{d^2}{dx^2} \{\frac{(N-x+1)\lambda + R\mu}{2}\ f_6(x)\}, \qquad R \le x \le N,$$

which are the homogeneous second-order linear differential equations.

It should be noted that equations 2.15 and 2.16 have the same form as the limiting case of the Fokker-Planck equation which is given by equation 1.9. Comparing equations 2.15 and 2.16 with equation 1.9, we obtain the quantities $A_{1i}(x)$ and $A_{2i}(x)$, $(i = 1, 2, \ldots, 6)$ which represent the diffusion parameters for the M/M/R MRP with spares in steady state. The steady-state diffusion parameters for the M/M/R MRP with spares are given in Table 2.5 for $R \le S$ and in Table 2.6 for $R > S$.

Table 2.5 The diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ for the M/M/R MRP with spares when $R \le S$.

i	Range of x	$A_{1i}(x)$	$A_{2i}(x)$
1	$0 \le x < R$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda + (S - x + 1)\alpha + (x + 1)\mu$
2	$R \le x \le S$	$M\lambda + (S - x)\alpha - R\mu$	$M\lambda + (S - x + 1)\alpha + R\mu$
3	$S < x \le N$	$(N - x)\lambda - R\mu$	$(N - x + 1)\lambda + R\mu$

Table 2.6 The diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ for the M/M/R MRP with spares when $R > S$.

i	Range of x	$A_{1i}(x)$	$A_{2i}(x)$
4	$0 \le x \le S$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda + (S - x + 1)\alpha + (x + 1)\mu$
5	$S < x < R$	$(N - x)\lambda - x\mu$	$(N - x + 1)\lambda + (x + 1)\mu$
6	$R \le x \le N$	$(N - x)\lambda - R\mu$	$(N - x + 1)\lambda + R\mu$

The respective solutions of equations 2.15 subject to the boundary condition specified by equation 1.13 are given by (see, e.g., Heyman and Sobel [29] for a similar solution involving one reflecting barrier)

$$(2.17) \qquad f_j(x) = \frac{K_j}{A_{2j}(x)} \exp\left\{ \int^{x} \frac{A_{1j}(y)}{A_{2j}(y)} dy \right\}, \quad \text{for } 0 \le x \le N, \text{ and}$$

$j = 1, 2, \ldots, 6$, where $K_j > 0$ is the constant of integration.

Using the results of Table 2.5 in equation 2.17, we obtain for $R \le S$ the following expressions for $f_1(x)$, $f_2(x)$, and $f_3(x)$ in terms of the dimensionless quantities $\theta_\lambda = \lambda/\mu$ and $\theta_\alpha = \alpha/\mu$:

$$(2.18a) \qquad f_1(x) = \frac{K_1}{M\lambda+(S-x+1)\alpha+(x+1)\mu} \exp\left\{ \int_0^{x} \frac{2[M\lambda+(S-y)\alpha-y\mu]}{M\lambda+(S-y+1)\alpha+(y+1)\mu} dy \right\}$$

$$= K_1[M\theta_\lambda+(S-x+1)\theta_\alpha+(x+1)]^{-1} \left[\frac{M\theta_\lambda+(S-x+1)\theta_\alpha+(x+1)}{M\theta_\lambda+(S+1)\theta_\alpha+1}\right]^{\beta_1}$$

$$* \exp\left\{\frac{2(\theta_\alpha+1)x}{\theta_\alpha-1}\right\} \qquad \text{if } \theta_\alpha \ne 1$$

$$\equiv K_1 g_1(x), \qquad \text{for } 0 \le x < R,$$

$$(2.18b) \quad f_2(x) = \frac{K_2}{M\lambda+(S-x+1)\alpha+R\mu} \exp\left\{ \int_R^{x} \frac{2[M\lambda+(S-y)\alpha-R\mu]}{M\lambda+(S-y+1)\alpha+R\mu} dy \right\}$$

$$= K_2[M\theta_\lambda+(S-x+1)\theta_\alpha+R]^{-1} \left[\frac{M\theta_\lambda+(S-x+1)\theta_\alpha+R}{M\theta_\lambda+(S-R+1)\theta_\alpha+R}\right]^{\beta_2} \exp\{2(x-R)\}$$

$$\equiv K_2 g_2(x), \qquad \text{for } R \le x \le S,$$

(2.18c) $$f_3(x) = \frac{K_3}{(N-x+1)\lambda+R\mu} \exp\{ \int_S^x \frac{2[(N-y)\lambda-R\mu]}{(N-y+1)\lambda+R\mu} dy\}$$

$$= K_3[(N-x+1)\theta_\lambda+R]^{-1} [\frac{(N-x+1)\theta_\lambda + R}{(M+1)\theta_\lambda + R}]^{\beta_3} \exp\{2(x-S)\}$$

$$= K_3 g_3(x), \quad \text{for } S < x \le N,$$

where

$$\beta_1 = [4(M\theta_\lambda + S\theta_\alpha) + 2(\theta_\alpha + 1)^2] / (\theta_\alpha - 1)^2,$$

$$\beta_2 = 2 + (4R / \theta_\alpha),$$

$$\beta_3 = 2 + (4R / \theta_\lambda).$$

Likewise for R > S, using the results of Table 2.6 in equation 2.17, we obtain the following expressions in terms of the dimensionless quantities $\theta_\lambda = \lambda/\mu$ and $\theta_\alpha = \alpha/\mu$:

(2.19a) $$f_4(x) = \frac{K_4}{M\lambda+(S-x+1)\alpha+(x+1)\mu} \exp\{ \int_0^x \frac{2[M\lambda+(S-y)\alpha-y\mu]}{M\lambda+(S-y+1)\alpha+(y+1)\mu} dy\}$$

$$= K_4[M\theta_\lambda+(S-x+1)\theta_\alpha+(x+1)]^{-1} [\frac{M\theta_\lambda+(S-x+1)\theta_\alpha+(x+1)}{M\theta_\lambda+(S+1)\theta_\alpha+1}]^{\beta_4}$$

$$* \exp\{\frac{2(\theta_\alpha+1)x}{\theta_\alpha-1}\} \qquad \text{if } \theta_\alpha \ne 1$$

$$= K_4 g_4(x), \quad \text{for } 0 \le x \le S,$$

$$(2.19b) \qquad f_5(x) = \frac{K_5}{(N-x+1)\lambda+(x+1)\mu} \exp\left\{ \int_S^x \frac{2[(N-y)\lambda-y\mu]}{(N-y+1)\lambda+(y+1)\mu} dy \right\}$$

$$= K_5[(N-x+1)\theta_\lambda+(x+1)]^{-1} \left[\frac{(N-x+1)\theta_\lambda+(x+1)}{(M+1)\theta_\lambda+(S+1)}\right]^{\beta_5}$$

$$* \exp\left\{\frac{2(\theta_\lambda+1)(x-S)}{\theta_\lambda - 1}\right\} \qquad \text{if } \theta_\lambda \neq 1$$

$$\equiv K_5 g_5(x), \quad \text{for } S < x \leq R,$$

$$(2.19c) \qquad f_6(x) = \frac{K_6}{(N-x+1)\lambda+R\mu} \exp\left\{ \int_R^x \frac{2[(N-y)\lambda-R\mu]}{(N-y+1)\lambda+R\mu} dy \right\}$$

$$= K_6[(N-x+1)\theta_\lambda+R]^{-1} \left[\frac{(N-x+1)\theta_\lambda+R}{(N-R+1)\theta_\lambda+R}\right]^{\beta_6} \exp\{\, 2(x-R) \,\}$$

$$\equiv K_6 g_6(x), \quad \text{for } R < x \leq N,$$

where

$$\beta_4 = [4(M\theta_\lambda + S\theta_\alpha) + 2(\theta_\alpha + 1)^2] \,/\, (\theta_\alpha - 1)^2,$$

$$\beta_5 = [4N\theta_\lambda + 2(\theta_\lambda + 1)^2] \,/\, (\theta_\lambda - 1)^2,$$

$$\beta_6 = 2 + (4R \,/\, \theta_\lambda).$$

Solutions $f_1(x)$, $f_2(x)$, and $f_3(x)$ can be verified to satisfy the boundary condition specified by equation 1.13. Equations 2.18a through 2.18c can be rewritten as

$$(2.20a) \qquad f_1(x) = K_1 g_1(x), \quad \text{for } 0 \leq x < R,$$

$$(2.20b) \qquad f_2(x) = K_2 g_2(x), \quad \text{for } R \leq x \leq S,$$

$$(2.20c) \qquad f_3(x) = K_3 g_3(x), \quad \text{for } S < x \leq N.$$

Similar type expressions can be written for equations 2.19a through 2.19c.

For $R \leq S$, the unknown constants K_1, K_2, and K_3 are determined by two conditions:

The first condition comes from the normalization criterion, namely,

$$(2.21) \qquad 1 = \int_0^N f(x)dx = K_1 \int_0^R g_1(x)dx + K_2 \int_R^S g_2(x)dx + K_3 \int_S^N g_3(x)dx.$$

The second condition assumes continuity of $f(x)$ at $x = R$ and $x = S$; i.e., $f_1(R) = f_2(R)$, and $f_2(S) = f_3(S)$. Therefore, we obtain

$$(2.22a) \qquad K_1 g_1(R) = K_2 g_2(R),$$

and

$$(2.22b) \qquad K_2 g_2(S) = K_3 g_3(S).$$

The unknown constants K_1, K_2, and K_3 are determined by solving equations 2.21 and 2.22 simultaneously (Burden and Faires [10]). Once the unknown constants K_1, K_2, and K_3 are obtained, $f(x)$ can be determined for the three different ranges of x. This will specify the p.d.f. $f(x)$, where x is the total number of failed machines in the system. The results for the expected number of failed machines, $E(X_1)$,

$$(2.23) \qquad E(X_1) = K_1 \int_0^R xg_1(x)dx + K_2 \int_R^S xg_2(x)dx + K_3 \int_S^N xg_3(x)dx$$

of the diffusion model for the M/M/R MRP with spares in steady-state are shown in Table 2.7. Equivalent results may be obtained when R > S shown in Table 2.8.

Table 2.7 The expected number of failed machines for the diffusion model for the M/M/R MRP with warm standbys in steady-state (M=10, S=5, N=15).

θ_α=0.05	R				
θ_λ	1	2	3	4	5
0.2	9.9514	5.0729	2.9587	2.4784	2.3687
0.4	12.3329	9.9481	7.4113	5.4488	4.5065
0.6	13.0595	11.5663	9.9500	8.3014	6.9227
0.8	13.4031	12.3329	11.1715	9.9563	8.7619
1.0	13.6019	12.7746	11.8769	10.9325	9.9680
1.2	13.7310	13.0595	12.3329	11.5665	10.7748
1.4	13.8215	13.2577	12.6500	12.0086	11.3428
1.6	13.8885	13.4031	12.8824	12.3329	11.7614
1.8	13.9399	13.5143	13.0595	12.5802	12.0813
2.0	13.9808	13.6019	13.1987	12.7746	12.3330

Table 2.7--Continued

θ_α=0.1	R				
θ_λ	1	2	3	4	5
0.2	9.9583	5.2439	3.1166	2.6003	2.4799
0.4	12.3329	9.9519	7.4456	5.5085	4.5663
0.6	13.0595	11.5664	9.9524	8.3119	6.9411
0.8	13.4031	12.3329	11.1717	9.9578	8.7660
1.0	13.6019	12.7746	11.8770	10.9328	9.9689
1.2	13.7310	13.0595	12.3329	11.5665	10.7750
1.4	13.8215	13.2577	12.6500	12.0086	11.3429
1.6	13.8885	13.4031	12.8824	12.3329	11.7614
1.8	13.9399	13.5143	13.0595	12.5802	12.0813
2.0	13.9808	13.6019	13.1987	12.7746	12.3330

Table 2.8 The expected number of failed machines for the diffusion model for the M/M/R MRP with warm standbys in steady-state (M=10, S=5, N=15).

θ_α=0.05	R				
θ_λ	6	7	8	9	10
0.2	2.3440	2.3394	2.3388	2.3388	2.3388
0.4	4.1566	4.0436	4.0119	4.0045	4.0032
0.6	6.0775	5.6771	5.5206	5.4692	5.4555
0.8	7.7696	7.1212	6.7865	6.6460	6.5980
1.0	9.0566	8.3231	7.8489	7.6041	7.5027
1.2	9.9880	9.2777	8.7373	8.4060	8.2444
1.4	10.6675	10.0200	9.4711	9.0856	8.8696
1.6	11.1763	10.5975	10.0717	9.6627	9.4050
1.8	11.5682	11.0521	10.5626	10.1519	9.8671
2.0	11.8778	11.4157	10.9654	10.5665	10.2676

Table 2.8--Continued

θ_α=0.1	R				
θ_λ	6	7	8	9	10
0.2	2.4529	2.4479	2.4472	2.4472	2.4472
0.4	4.2132	4.0987	4.0665	4.0590	4.0576
0.6	6.0985	5.6983	5.5415	5.4901	5.4763
0.8	7.7758	7.1283	6.7938	6.6533	6.6053
1.0	9.0582	8.3254	7.8514	7.6067	7.5052
1.2	9.9885	9.2784	8.7381	8.4069	8.2453
1.4	10.6677	10.0202	9.4714	9.0860	8.8699
1.6	11.1764	10.5976	10.0718	9.6628	9.4051
1.8	11.5682	11.0522	10.5626	10.1520	9.8672
2.0	11.8778	11.4157	11.9654	10.5666	10.2676

2.2 Approximation to the M/M/R Model

The above approach suggests that an appropriate choice of $A_{1i}(x)$ and $A_{2i}(x)$ ($i = 1, 2, \ldots, 6$) may be used to provide an approximation for the steady-state probability density functions of the number of failed machines in the more general G/G/R MRP with warm standbys in Chapter 3. To do this we develop a heuristic derivation for the infinitesimal mean and variance for the M/M/R queue length over small time interval dt.

2.2.1 The Infinitesimal Mean and Variance for the M/M/R Queue Length over Time Interval dt

Let

$M(t) \equiv$ the number of failed operating machines during the interval $(0, t)$,

$S(t) \equiv$ the number of failed spare machines during the interval $(0, t)$,

$R(t) \equiv$ the total number of repair completions during the interval $(0, t)$,

$N(t) \equiv$ the number of failed machines in the system at time t.

N(t) represents the total number of machines which either are being repaired by one of the repairmen or are waiting for repair in the queue at time t. N(t) is given by

$$N(t) = N(0) + M(t) + S(t) - R(t), \tag{2.24}$$

also

$$N(t+dt) = N(0) + M(t+dt) + S(t+dt) - R(t+dt). \tag{2.25}$$

During the time interval (t, t+dt), the number of failed machines in the system changes by the number of arrivals (breakdowns of operating

and spare machines) minus the number of repair completions. Thus we obtain

$$(2.26) \qquad N(t+dt) - N(t) = [M(t+dt) - M(t)] + [S(t+dt) - S(t)] - [R(t+dt) - R(t)].$$

If at time t there are n failed machines (i.e., $N(t) = n$), we will specify that the queueing system is in state n. Then the random variables of the number of operating machines at time t, and of the number of available spare machines at time t will be specified. For example, given $N(t) = n$ and $0 \le n < R \le S$, we observe that a transition from state n to state n + 1 is caused by the next breakdown of one among the M operating machines or by the next breakdown of one among the (S - n) spare machines. The probability that the next operating machine breaks down during the interval (t, t+dt) is $M\lambda dt + o(dt)$, namely,

$$(2.27) \qquad P\{M(t+dt) - M(t) = 1 \mid N(t) = n\} = M\lambda dt + o(dt),$$

where o(dt) is a function such that $\lim_{dt \to 0} [o(dt)/dt] = 0$.

Meanwhile, there are (S - n) spare machines available at time t. The probability that the next spare machine breaks down during the interval (t, t+dt) is $(S - n)\alpha dt + o(dt)$, namely,

$$(2.28) \qquad P\{S(t+dt) - S(t) = 1 \mid N(t) = n\} = (S - n)\alpha dt + o(dt).$$

We observe that in the case when $0 \le n < R$ if the queueing system is in state n, then n failed machines are being repaired and (R - n) repairmen are idle. It is clear that a transition from state n to state n - 1 occurs when a machine is repaired and returned to a standby state. The probability that during the interval (t, t+dt), n failed machines are repaired and returned to a standby state is

(2.29) $\quad P\{R(t+dt) - R(t) = 1 \mid N(t) = n\} = n\mu dt + o(dt).$

Following the above procedures and given $N(t) = n$, we obtain the conditional probabilities of $M(t+dt) - M(t) = 1$, $S(t+dt) - S(t) = 1$, and $R(t+dt) - R(t) = 1$ for three different ranges of n shown in Table 2.9 when $R \le S$ and in Table 2.10 when $R > S$.

Table 2.9 The conditional probabilities of $M(t+dt) - M(t) = 1$, $S(t+dt) - S(t) = 1$, and $R(t+dt) - R(t) = 1$, given $N(t) = n$ and when $R \le S$.

Conditional Probability	$0 \le n < R$	$R \le n \le S$	$S < n \le N$
$P\{M(t+dt)-M(t)=1 \mid N(t)=n\}$	$M\lambda dt+o(dt)$	$M\lambda dt+o(dt)$	$(N-n)\lambda dt+o(dt)$
$P\{S(t+dt)-S(t)=1 \mid N(t)=n\}$	$(S-n)\alpha dt+o(dt)$	$(S-n)\alpha dt+o(dt)$	0
$P\{R(t+dt)-R(t)=1 \mid N(t)=n\}$	$n\mu dt+o(dt)$	$R\mu dt+o(dt)$	$R\mu dt+o(dt)$

Table 2.10 The conditional probabilities of $M(t+dt) - M(t) = 1$, $S(t+dt) - S(t) = 1$, and $R(t+dt) - R(t) = 1$, given $N(t) = n$ and when $R > S$.

Conditional Probability	$0 \le n \le S$	$S < n \le R$	$R < n \le N$
$P\{M(t+dt)-M(t)=1 \mid N(t)=n\}$	$M\lambda dt+o(dt)$	$(N-n)\lambda dt+o(dt)$	$(N-n)\lambda dt+o(dt)$
$P\{S(t+dt)-S(t)=1 \mid N(t)=n\}$	$(S-n)\alpha dt+o(dt)$	0	0
$P\{R(t+dt)-R(t)=1 \mid N(t)=n\}$	$n\mu dt+o(dt)$	$n\mu dt+o(dt)$	$R\mu dt+o(dt)$

Since

$$P\{M(t+dt)-M(t) = 0 \mid N(t) = n\} = 1 - P\{M(t+dt)-M(t)=1 \mid N(t) = n\},$$

(2.30) $$P\{S(t+dt)-S(t) = 0 \mid N(t) = n\} = 1 - P\{S(t+dt)-S(t)=1 \mid N(t) = n\},$$

$$P\{R(t+dt)-R(t) = 0 \mid N(t) = n\} = 1 - P\{R(t+dt)-R(t)=1 \mid N(t) = n\},$$

therefore from Table 2.9 we obtain the conditional probabilities of $M(t+dt) - M(t) = 0$, $S(t+dt) - S(t) = 0$, and $R(t+dt) - R(t) = 0$, given $N(t) = n$ and when $R \leq S$ as follows:

(i) $0 \leq n < R$

$$
\begin{aligned}
&P\{M(t+dt)-M(t) = 0 \mid N(t) = n\} = 1 - [M\lambda dt + o(dt)], \\
(2.31)\qquad &P\{S(t+dt)-S(t) = 0 \mid N(t) = n\} = 1 - [(S - n)\alpha dt + o(dt)], \\
&P\{R(t+dt)-R(t) = 0 \mid N(t) = n\} = 1 - [n\mu dt + o(dt)],
\end{aligned}
$$

(ii) $R \leq n \leq S$

$$
\begin{aligned}
&P\{M(t+dt)-M(t) = 0 \mid N(t) = n\} = 1 - [M\lambda dt + o(dt)], \\
(2.32)\qquad &P\{S(t+dt)-S(t) = 0 \mid N(t) = n\} = 1 - [(S - n)\alpha dt + o(dt)], \\
&P\{R(t+dt)-R(t) = 0 \mid N(t) = n\} = 1 - [R\mu dt + o(dt)],
\end{aligned}
$$

(iii) $S < n \leq N$

$$
\begin{aligned}
&P\{M(t+dt)-M(t) = 0 \mid N(t) = n\} = 1 - [(N - n)\lambda dt + o(dt)], \\
(2.33)\qquad &P\{S(t+dt)-S(t) = 0 \mid N(t) = n\} = 1, \\
&P\{R(t+dt)-R(t) = 0 \mid N(t) = n\} = 1 - [R\mu dt + o(dt)].
\end{aligned}
$$

From Table 2.10, equivalent results may be obtained when $R > S$.

We list the relationship between $N(t+dt) - N(t)$ and $M(t+dt) - M(t)$, $S(t+dt) - S(t)$, and $R(t+dt) - R(t)$ given in Table 2.11.

Table 2.11 Relationship between N(t+dt) - N(t) and M(t+dt) - M(t), S(t+dt) - S(t), R(t+dt) - R(t).

	M(t+dt)-M(t) =	S(t+dt)-S(t)=	R(t+dt)-R(t) =
N(t+dt)-N(t) = -1	0	0	1
N(t+dt)-N(t) = 0	0 1 0	0 0 1	0 1 1
N(t+dt)-N(t) = 1	1 0	0 1	0 0

When $R \leq S$, using equations 2.31 through 2.33 and the results of Table 2.9 and Table 2.11, we obtain the conditional probabilities of $N(t+dt)-N(t) = -1$, $N(t+dt)-N(t) = 0$, and $N(t+dt)-N(t) = 1$, given $N(t) = n$ as follows:

(i) for $0 \leq n < R$

(2.34) $$P\{N(t+dt)-N(t) = -1 | N(t) = n\}$$
$$= P\{M(t+dt)-M(t) = 0|N(t)=n\} \bullet P\{S(t+dt)-S(t) = 0|N(t)=n\}$$
$$\bullet\ P\{R(t+dt)-R(t) = 1|N(t)=n\}$$
$$= n\mu dt + o(dt),$$

(2.35) $$P\{N(t+dt)-N(t) = 0|N(t) = n\}$$
$$= P\{M(t+dt)-M(t) = 0|N(t)=n\} \bullet P\{S(t+dt)-S(t) = 0|N(t)=n\}$$
$$\bullet\ P\{R(t+dt)-R(t) = 0|N(t)=n\} + o(dt)$$
$$= 1 - [M\lambda + (S - n)\alpha + n\mu]dt + o(dt),$$

$$(2.36)\quad P\{N(t+dt)-N(t) = 1|N(t) = n\}$$

$$= P\{M(t+dt)-M(t) = 1|N(t)=n\} \cdot P\{S(t+dt)-S(t) = 0|N(t)=n\}$$

$$\cdot P\{R(t+dt)-R(t) = 0|N(t)=n\}$$

$$+ P\{M(t+dt)-M(t) = 0|N(t)=n\} \cdot P\{S(t+dt)-S(t) = 1|N(t)=n\}$$

$$\cdot P\{R(t+dt)-R(t) = 0|N(t)=n\}$$

$$= [M\lambda + (S - n)\alpha]dt + o(dt).$$

Similar, we obtain

(ii) for $R \le n \le S$

$$(2.37)\quad P\{N(t+dt)-N(t) = -1|N(t) = n\} = R\mu dt + o(dt),$$

$$(2.38)\quad P\{N(t+dt)-N(t) = 0|N(t) = n\} = 1 - [M\lambda + (S-n)\alpha + R\mu]dt + o(dt),$$

$$(2.39)\quad P\{N(t+dt)-N(t) = 1|N(t) = n\} = [M\lambda + (S-n)\alpha]dt + o(dt),$$

and (iii) for $S < n \le N$

$$(2.40)\quad P\{N(t+dt)-N(t) = -1|N(t) = n\} = R\mu dt + o(dt),$$

$$(2.41)\quad P\{N(t+dt)-N(t) = 0|N(t) = n\} = 1 - [(N-n)\lambda + R\mu]dt + o(dt),$$

$$(2.42)\quad P\{N(t+dt)-N(t) = 1|N(t) = n\} = (N-n)\lambda dt + o(dt).$$

Likewise, when $R > S$ we obtain the equivalent results for the conditional probabilities of $N(t+dt)-N(t) = -1$, $N(t+dt)-N(t) = 0$, and $N(t+dt)-N(t) = 1$, given $N(t) = n$ as follows:

(i) for $0 \le n \le S$

$$(2.43)\quad P\{N(t+dt)-N(t) = -1|N(t) = n\} = n\mu dt + o(dt),$$

$$(2.44)\quad P\{N(t+dt)-N(t) = 0|N(t) = n\} = 1 - [M\lambda + (S-n)\alpha + n\mu]dt + o(dt),$$

$$(2.45)\quad P\{N(t+dt)-N(t) = 1|N(t) = n\} = [M\lambda + (S - n)\alpha]dt + o(dt),$$

(ii) for $S < n \le R$

(2.46) $P\{N(t+dt)-N(t) = -1 | N(t) = n\} = n\mu dt + o(dt)$,

(2.47) $P\{N(t+dt)-N(t) = 0 | N(t) = n\} = 1 - [(N - n)\lambda + n\mu]dt + o(dt)$,

(2.48) $P\{N(t+dt)-N(t) = 1 | N(t) = n\} = (N - n)\lambda dt + o(dt)$,

(iii) for $R < n \le N$

(2.49) $P\{N(t+dt)-N(t) = -1 | N(t) = n\} = R\mu dt + o(dt)$,

(2.50) $P\{N(t+dt)-N(t) = 0 | N(t) = n\} = 1 - [(N-n)\lambda + R\mu]dt + o(dt)$,

(2.51) $P\{N(t+dt)-N(t) = 1 | N(t) = n\} = (N-n)\lambda dt + o(dt)$.

As proved in Appendix B, we obtain the conditional expectations and the conditional variances of $N(t+dt) - N(t)$, given $N(t) = n$ and $N(0) = 0$ for the M/M/R MRP with spares shown in Table 2.12 when $R \le S$. Following the procedures given in Appendix B, we obtain the conditional expectations and the conditional variances of $N(t+dt) - N(t)$, given $N(t) = n$ for the M/M/R MRP with spares shown in Table 2.13 when $R > S$.

Table 2.12 The conditional expectations and the conditional variances of $N(t+dt) - N(t)$, given $N(t) = n$ for the M/M/R MRP with spares when $R \le S$.

Range of n	$E[N(t+dt) - N(t) \| N(t)=n]$	$Var[N(t+dt) - N(t) \| N(t)=n]$
$0 \le n < R$	$[M\lambda + (S-n)\alpha - n\mu]dt + o(dt)$	$[M\lambda + (S-n)\alpha + n\mu]dt + o(dt)$
$R \le n \le S$	$[M\lambda + (S-n)\alpha - R\mu]dt + o(dt)$	$[M\lambda + (S-n)\alpha + R\mu]dt + o(dt)$
$S < n \le N$	$[(N-n)\lambda - R\mu]dt + o(dt)$	$[(N-n)\lambda + R\mu]dt + o(dt)$

Table 2.13 The conditional expectations and the conditional variances of $N(t+dt) - N(t)$, given $N(t) = n$ for the M/M/R MRP with spares when $R > S$.

Range of n	$E[N(t+dt) - N(t) \mid N(t)=n]$	$Var[N(t+dt) - N(t) \mid N(t)=n]$
$0 \le n \le S$	$[M\lambda + (S-n)\alpha - n\mu]dt + o(dt)$	$[M\lambda + (S-n)\alpha + n\mu]dt + o(dt)$
$S < n \le R$	$[(N-n)\lambda - n\mu]dt + o(dt)$	$[(N-n)\lambda + n\mu]dt + o(dt)$
$R < n \le N$	$[(N-n)\lambda - R\mu]dt + o(dt)$	$[(N-n)\lambda + R\mu]dt + o(dt)$

As proved in Appendix C, we obtain

$$(2.52) \qquad \begin{aligned} E[N(t+dt) - N(t) \mid N(t) = n] = {} & E[M(t+dt) - M(t) \mid N(t) = n] \\ & + E[S(t+dt) - S(t) \mid N(t) = n] \\ & - E[R(t+dt) - R(t) \mid N(t) = n], \end{aligned}$$

and

$$(2.53) \qquad \begin{aligned} Var[N(t+dt) - N(t) \mid N(t) = n] = {} & Var[M(t+dt) - M(t) \mid N(t) = n] \\ & + Var[S(t+dt) - S(t) \mid N(t) = n] \\ & + Var[R(t+dt) - R(t) \mid N(t) = n]. \end{aligned}$$

In the diffusion approximation methodology, the discrete variable n is approximated by a continuous variable x, and the discrete process $\{N(t), t \ge 0\}$ is approximated by a diffusion process $\{X(t), t \ge 0\}$ with the same infinitesimal mean $A_i(x)$ and infinitesimal variance $B_i(x)$ which are determined by

$$(2.54) \qquad A_i(x) = \lim_{dt \to 0} \frac{E[X(t+dt) - X(t) \mid X(t) = x]}{dt},$$

$$(2.55) \qquad B_i(x) = \lim_{dt \to 0} \frac{Var[X(t+dt) - X(t) \mid X(t) = x]}{dt}.$$

Using the results of Table 2.12 and Table 2.13 in equations 2.45, we obtain the infinitesimal mean $A_i(x)$ and the infinitesimal variance $B_i(x)$ for the M/M/R queue length $X(t)$ shown in Table 2.14 when $R \le S$, and in Table 2.15 when $R > S$, respectively.

Table 2.14 The infinitesimal mean $A_i(x)$ and the infinitesimal variance $B_i(x)$ for the M/M/R queue length $X(t)$ when $R \le S$.

i	Range of x	$A_i(x)$	$B_i(x)$
1	$0 \le x < R$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda + (S - x)\alpha + x\mu$
2	$R \le x \le S$	$M\lambda + (S - x)\alpha - R\mu$	$M\lambda + (S - x)\alpha + R\mu$
3	$S < x \le N$	$(N - x)\lambda - R\mu$	$(N - x)\lambda + R\mu$

Table 2.15 The infinitesimal mean $A_i(x)$ and the infinitesimal variance $B_i(x)$ for the M/M/R queue length $X(t)$ when $R > S$.

i	Range of x	$A_i(x)$	$B_i(x)$
4	$0 \le x \le S$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda + (S - x)\alpha + x\mu$
5	$S < x \le R$	$(N - x)\lambda - x\mu$	$(N - x)\lambda + x\mu$
6	$R < x \le N$	$(N - x)\lambda - R\mu$	$(N - x)\lambda + R\mu$

Comparing Table 2.14 and Table 2.15 with Table 2.5 and Table 2.6, respectively. We note that $A_{1i}(x)$ corresponds to the infinitesimal mean $A_i(x)$ of $X(t)$ while $A_{2i}(x)$ corresponds approximately to the infinitesimal variance $B_i(x)$ of $X(t)$, for $i = 1, 2, \ldots, 6$. We may suggest that an appropriate choice of $A_i(x)$ and $B_i(x)$ will provide a good approximation to the queue length of the G/G/R MRP.

CHAPTER 3

THE G/G/R MACHINE REPAIR PROBLEM WITH WARM STANDBYS

In this chapter, we use renewal theory and apply the theory of diffusion approximation associated with heavy traffic condition to study some of the characteristics of the G/G/R machine repair problem (MRP) with warm standby spares. Exact and tractable steady-state solutions for the more general G/G/R model are unknown. However, one can use diffusion approximation method to approximate a G/G/R system or a G/G/R MRP with spares. This becomes particularly useful when the coefficient of variation of the service time distribution, or interarrival time distribution, or interarrival time and service time distributions is not unity.

Several authors have used diffusion approximation to investigate the G/G/R system and the G/G/R MRP with spares. This includes Iglehart [31], Kobayashi [41], Heyman [28], Sunaga et al. [58], Halachmi and Franta [25], Whitt [72], Karmeshu and Jaiswal [33], Haryono and Sivazlian [27], Sivazlian and Wang [55], and Gelenbe and Pujolle [20]. The methodology of this chapter follows Halachmi and Franta's approach [25].

3.1 Approximation to the G/G/R Model

The approximate infinitesimal mean and infinitesimal variance (or diffusion parameters) of the diffusion equations are obtained (i) under the assumption that the input characteristics of the problem are defined only by their first two moments rather than their probability distribution function, (ii) under the assumption of heavy traffic approximation, that is, when queues of failed machines in the repair stage are almost always non-empty, and (iii) using well-known asymptotic results from renewal theory.

We attempt now to obtain approximate expressions for diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ for the G/G/R MRP using basic concepts of renewal theory. We recall that if $\{Y(t);\ t \geq 0\}$ is an ordinary renewal process, then for large t, the mean and variance of Y(t) is given by

$$(3.1) \qquad E[Y(t)] \approx \frac{t}{\nu} \quad \text{and} \quad Var[Y(t)] \approx \frac{\sigma^2 t}{\nu^3},$$

where ν and $\sigma 2$ are the mean and variance of the times between renewals.

3.1.1 The Expectation and Variance for the G/G/R Queue Length N(t) for Large Value of t

For a G/G/R queue, let

(i) λ^{-1} and σ_M^2 be the mean and variance respectively of the succession of uptimes for the operating machines.

(ii) α^{-1} and σ_S^2 be the mean and variance respectively of the succession of uptimes for the spare machines.

(iii) μ^{-1} and σ_R^2 be the mean and variance respectively of the succession of repair times assumed to be the same at each repair station.

We assume that the succession of uptimes and repair times form a sequence of independent and identically distributed random variables. As long as there is at least one spare machine which is not in a failed state, the replacement of failed operating machines constitutes a renewal process. More specifically, since there are M operating machines, we can consider the operating machines to be made of M states, for example, each state corresponding to the location of an operating machine. When the machine in the i-th state fails, then as long as there is a working spare machine, it is replaced by a machine as good as new. This is true as long as $n \leq S$, where n is a random variable denoting the number of failed machines in the repair stage. In this case, the total number of arrivals due to failed operating machines from all states is given by

$$M(t) = \sum_{i=1}^{M} M_i(t), \tag{3.2}$$

where $M_i(t)$, i = 1, 2, ..., M, is the number of failed operating machines at the i-state during the interval (0, t).

Each of the $M_i(t)$'s is assumed to be an ordinary renewal process and the $M_i(t)$'s are assumed to be independent and identically distributed random variables. Then M(t) is the superposition of M independent renewal process, and for large t, we have

$$E[M(t)] = \sum_{i=1}^{M} E[M_i(t)] \approx M\lambda t,$$

and

$$\text{Var}[M(t)] = \sum_{i=1}^{M} \text{Var}[M_i(t)] \approx M\lambda^3 \sigma_M^2 t, \quad \text{for } 0 \le n < R.$$

Similarly, we obtain

$$E[M(t)] \approx M\lambda t \quad \text{and} \quad \text{Var}[M(t)] \approx M\lambda^3 \sigma_M^2 t, \quad \text{for } R \le n \le S.$$

This results will be discussed in the sequel.

When the system is short, i.e., there are less than M operating machines ($n > S$) and all spares are used, the number of available operating machines at any instant of time t will be a random variable. This is due to either one of the following reasons:

(i) Any one of the operating machines may fail, thus joining the waiting line for repair and this will cause a reduction in the number of operating machines.

(ii) A repair may be completed and the repaired machine will join one of the free states of the operating machines, thus causing an increas in the number of operating machines.

Since, at this point in time, no spare machines are available when an operating machine fails, the replacement process for the i-th state is not a renewal process. This is due to the fact that the only machines available for replacement are in the repair stage and that no repair may have been completed at the time of failure at the i-th state.

Recall that N(t) is the number of failed machines in the system at time t. In order to obtain approximate expressions for E[N(t)] and Var[N(t)] when the system is short ($n > S$), it is necessary to make

certain assumptions which rely on the fact that one is operating under heavy traffic condition. This corresponds to the case when the probability of the system being empty, that is of no failed machines in the system, is for all practical purposes zero. The justification of this assumption using theoretically rigorous arguments is far from being easy. We do provide however a heuristic argument of the conditions prevailing in the repair stage of the system to support to a certain extent, the prevalence of this condition. Let

$M_i(t) \equiv$ the number of failed operating machines at the i-th state during the interval $(0, t)$,

$S_j(t) \equiv$ the number of failed spare machines at the j-th state during the interval $(0, t)$,

$R_k(t) \equiv$ the number of repair completions at the k-th server during the interval $(0, t)$.

Thus we obtain

$$\text{(3.3)} \qquad N(t) = N(0) + \sum_{i=1}^{M} M_i(t) + \sum_{j=1}^{S} S_j(t) - \sum_{k=1}^{R} R_k(t)$$

$$= N(0) + M(t) + S(t) - R(t),$$

where $N(0)$ is the number of failed machines in the system at time $t = 0$, and

$$M(t) = \sum_{i=1}^{M} M_i(t), \; S(t) = \sum_{j=1}^{S} S_j(t), \text{ and } R(t) = \sum_{k=1}^{R} R_k(t).$$

Note that

$M(t) \equiv$ the total number of failed operating machines from all states during the interval $(0, t)$,

$S(t) \equiv$ the total number of failed spare machines from all states during the interval $(0, t)$,

$R(t) \equiv$ the total number of repair completions from all servers during the interval $(0, t)$.

In general $M_i(t)$, $S_j(t)$ and $R_k(t)$ are not independent, but under heavy traffic condition, the departure process is approximately independent of the arrival process. The assumption about the independence of $M_i(t)$, $S_j(t)$, and $R_k(t)$ is one that is in accordance with the point of view of several authors such as Kleinrock [40], Heyman [28], Gelenbe and Pujolle [20]. The ultimate criterion justifying the validity of this assumption is to show that for all practical purposes the system has a low probability of being empty. In our situations, it will be shown later on, with numerical examples, that even for moderate values of M, S, and R, the probability of the system having at least one failed machines is for all practical purposes unity, which corresponds exactly to the assumption of heavy traffic case.

We provide a heuristic argument for obtaining expressions for

(a) $E[M(t)]$ and $Var[M(t)]$, when $0 \le n < S$ and $S \le n \le N$,

(b) $E[S(t)]$ and $Var[S(t)]$, when $0 \le n < S$ and $S \le n \le N$,

(c) $E[R(t)]$ and $Var[R(t)]$, when $0 \le n < R$ and $R \le n \le N$.

(a) Expressions for $E[M(t)]$ and $Var[M(t)]$ when $0 \le n < S$ and $S \le n \le N$

For $0 \le n < S$, the spare machines are available when an operating machine fails. By hypothesis, $M_i(t)$, $1 \le i \le M$, is a renewal process, so that we can apply renewal limiting theorems to approximate the mean and variance of the number of failed operating machines at the i-th state during the interval $(0, t)$ as t is large. From equation 3.1, we obtain

$$E[M_i(t)] \approx \lambda t \quad \text{and} \quad Var[M_i(t)] \approx \lambda^3 \sigma_M^2 t, \quad 1 \le i \le M.$$

Since there are M states each corresponding to an operating machine and since the $M_i(t)$'s are assumed to be independent and identically distributed random variables, then M(t) is the superposition of M independent renewal process, and we have for large t

$$(3.4) \qquad E[M(t)] \approx M\lambda t \quad \text{and} \quad Var[M(t)] \approx M\lambda^3 \sigma_M^2 t.$$

For $\underline{S \le n \le N}$, the system is short, i.e., all spares are used and there are less than M operating machines. The system requires a minimum of one machine in operation to function properly. The number of operating machine is (N - n). The total number of operating machine states that are occupied is (N - n) and the number of free states is [M - (N - n)]. As soon as a repair is completed, the repaired machine joins one of the free states and becomes an operating machine. The breakdown process at any one of the occupied states is clearly not a renewal process. This is due to the fact that a failed operating machine is not immediately replaced by a repaired machine. We assume that a repair machine joins the states of the operating machines on the basis of first failed first replaced. Gaps may exist between the epoch of a failed machine and the epoch when it is replaced by a repaired machine. These gaps become less significant under heavy traffic condition which will be the case since all servers are busy and if the service rates of the repairman are not too small. Under these conditions, failures of the operating machines at the i-th state may be approximated by a renewal process. So that for large t, we obtain

$$E[M_i(t)] \approx \lambda t \quad \text{and} \quad Var[M_i(t)] \approx \lambda^3 \sigma_M^2 t, \quad 1 \le i \le N - n,$$

which implies that

$$(3.5) \qquad E[M(t)] = E[\sum_{i=1}^{N-n} M_i(t)] \approx (N - n)\lambda t,$$

and

$$(3.6) \qquad Var[M(t)] \approx (N - n)\lambda^3 \sigma_M^2 t.$$

(b) Expressions for $E[S(t)]$ and $Var[S(t)]$ when $0 \le n < S$ and $S \le n \le N$

For $0 \le n < S$, n of the spare machines will be operating and $(S - n)$ spares will be available. Recall that the spare machines consist of S states and $S_j(t)$ is the number of failed spare machines at the j-th state during the interval (0, t). Let

$S(t) = \sum_{j=n+1}^{S} S_j(t)$. Obviously, S(t) is not a renewal process since the only machines available for replacement are in the repair stage, and in general, no repair may be completed at the time of failure at the j-th state $(n + 1 \le j \le S)$.

Also when $0 \le n < S$, there are less than S machines in the repair stage, i.e., the number of failed machines in the repair stage is Min (S, n). The heavy traffic condition for this case is weaker than the case when there are more than S machines in the repair stage, $(S < n \le N)$, i.e., the number of failed machines in the repair stage is Max (S, n). In this case, we make the strong assumption that we can apply renewal limiting theorems to $S_j(t)$, so that for large t, we obtain

$$(3.7) \qquad E[S_j(t)] \approx \alpha t \quad \text{and} \quad Var[S_j(t)] \approx \alpha^3 \sigma_S^2 t, \quad n + 1 \le j \le S.$$

Since the $(S - n)$ spare machines form $(S - n)$ states and the $S_j(t)$'s are

assumed to be independent and identically distributed random variables. Then from equation 3.7 for large t, we obtain

$$(3.8) \qquad E[S(t)] = E[\sum_{j=n+1}^{S} S_j(t)]$$

$$\approx (S - n)\alpha t,$$

and

$$(3.9) \qquad Var[S(t)] \approx (S - n)\alpha^3 \sigma_s^2 t.$$

For $\underline{S < n \leq N}$, all spares are in the operating states; there are no spares available, hence

$$(3.10) \qquad E[S(t)] \approx 0,$$

and

$$(3.11) \qquad Var[S(t)] \approx 0.$$

(c) <u>Expressions for E[R(t)] and Var[R(t)] when $0 \leq n < R$ and $R \leq n < N$</u>

For $\underline{R \leq n \leq N}$, all R repairmen are busy, and each of the repairmen will serve all the time, with no gap between successive repairmen. Heavy traffic conditions prevail and it is reasonable to apply renewal limiting theorems to the departure process at each of the servers. Recall that $R_K(t)$ is the number of repair completions at the k-th repairman, $1 \leq k \leq R$, during the interval (0, t), then for large t we obtain

$$E[R_K(t)] \approx \mu t \quad \text{and} \quad Var[R_K(t)] \approx \mu^3 \sigma_R^2 t, \quad 1 \leq k \leq R.$$

Since the servers are assumed to act independently of each other, the $R_K(t)$'s, k = 1, 2, ..., R, are independently and identically distributed random variables. For large t, the approximate values of

the mean and variance of R(t), where $R(t) = \sum_{k=1}^{R} R_K(t)$, are given by

$$(3.12)\qquad E[R(t)] \approx R\mu t \quad \text{and} \quad Var[R(t)] \approx R\mu^3\sigma_R^2 t.$$

For <u>$0 \le n < R$</u>, n repairmen are busy and (R - n) repairmen are idle. In this case, heavy traffic conditions do not prevail because the total number of busy repairmen is less than R. We assume that the set of n busy repairmen is drawn from a continuum of R repairmen, each of which contributes in an additive way to the mean and variance of the departure process R(t). This suggests that we approximate the mean and variance of R(t), for large t, by

$$(3.13)\qquad E[R(t)] \approx n\mu t \quad \text{and} \quad Var[R(t)] \approx n\mu^3\sigma_R^2 t.$$

The above results can be summarized in Table 3.1 when $R \le S$ and in Table 3.2 when $R > S$.

Table 3.1 The approximate values of the mean and variance of M(t), S(t), and R(t) for large t, and when $R \le S$.

Range of n	E[M(t)]	E[S(t)]	E[R(t)]	Var[M(t)]	Var[S(t)]	Var[R(t)]
$0 \le n < R$	$M\lambda t$	$(S-n)\alpha t$	$n\mu t$	$M\lambda^3\sigma_M^2 t$	$(S-n)\alpha^3\sigma_S^2 t$	$n\mu^3\sigma_R^2 t$
$R \le n \le S$	$M\lambda t$	$(S-n)\alpha t$	$R\mu t$	$M\lambda^3\sigma_M^2 t$	$(S-n)\alpha^3\sigma_S^2 t$	$R\mu^3\sigma_R^2 t$
$S < n \le N$	$(N-n)\lambda t$	0	$R\mu t$	$(N-n)\lambda^3\sigma_M^2 t$	0	$R\mu^3\sigma_R^2 t$

Table 3.2 The approximate values of the mean and variance of $M(t)$, $S(t)$, and $R(t)$ for large t, and when $R > S$.

Range of n	E[M(t)]	E[S(t)]	E[R(t)]	Var[M(t)]	Var[S(t)]	Var[R(t)]
$0 \le n \le S$	$M\lambda t$	$(S-n)\alpha t$	$n\mu t$	$M\lambda^3\sigma_M{}^2 t$	$(S-n)\alpha^3\sigma_S{}^2 t$	$n\mu^3\sigma_R{}^2 t$
$S < n \le R$	$(N-n)\lambda t$	0	$n\mu t$	$(N-n)\lambda^3\sigma_M{}^2 t$	0	$n\mu^3\sigma_R{}^2 t$
$R < n \le N$	$(N-n)\lambda t$	0	$R\mu t$	$(N-n)\lambda^3\sigma_M{}^2 t$	0	$R\mu^3\sigma_R{}^2 t$

Let $C_M = \lambda^2\sigma_M^2$, $C_S = \alpha^2\sigma_S^2$, and $C_R = \mu^2\sigma_R^2$, where C_M, C_S, and C_R are the square coefficients of variation of the succession of the uptimes of the operating machines, the uptimes of the spare machines, and the repair times, respectively. We can express the processes $M(t)$, $S(t)$, and $R(t)$ as three independent normal processes, respectively. Therefore from Table 3.1 and Table 3.2, for large t, we obtain $M(t)$, $S(t)$, and $R(t)$ which are converged in normal distribution given in Table 3.3 for $R \le S$ and Table 3.4 for $R > S$.

Table 3.3 $M(t)$, $S(t)$, and $R(t)$ converge in normal distribution for large t, and for $R \le S$.

Range of n	$M(t) \xrightarrow{C}$	$S(t) \xrightarrow{C}$	$R(t) \xrightarrow{C}$
$0 \le n < R$	$N(M\lambda t,\ M\lambda C_M t)$	$N((S-n)\alpha t,\ (S-n)\alpha C_S t)$	$N(n\mu t,\ n\mu C_R t)$
$R \le n \le S$	$N(M\lambda t,\ M\lambda C_M t)$	$N((S-n)\alpha t,\ (S-n)\alpha C_S t)$	$N(R\mu t,\ R\mu C_R t)$
$S < n \le N$	$N((N-n)\lambda t,\ (N-n)\lambda C_M t)$	$N(0,\ 0)$	$N(R\mu t,\ R\mu C_R t)$

Table 3.4 $M(t)$, $S(t)$, and $R(t)$ converge in normal distribution for large t, and for $R > S$.

Range of n	$M(t) \xrightarrow{C}$	$S(t) \xrightarrow{C}$	$R(t) \xrightarrow{C}$
$0 \le n \le S$	$N(M\lambda t,\ M\lambda C_M t)$	$N((S-n)\alpha t,\ (S-n)\alpha C_S t)$	$N(n\mu t,\ n\mu C_R t)$
$S < n \le R$	$N(M\lambda t,\ M\lambda C_M t)$	$N((S-n)\alpha t,\ (S-n)\alpha C_S t)$	$N(R\mu t,\ R\mu C_R t)$
$R < n \le N$	$N((N-n)\lambda t,\ (N-n)\lambda C_M t)$	$N(0,\ 0)$	$N(R\mu t,\ R\mu C_R t)$

Note that $\xrightarrow{C}$ denotes convergence in distribution, and $N(\nu,\ \sigma^2)$ represents the normal distribution with mean ν and variance σ^2.

3.1.2 Approximate Expressions for the Diffusion Parameters in the G/G/R System for Steady-State

From equation 3.3, $N(t)$ is approximately a linear combination of the three independent normal processes $M(t)$, $S(t)$ and $R(t)$. Therefore $\{N(t),\ t \ge 0\}$ is approximately a normal process with mean and variance given respectively by the following: (Assuming $N(0) = 0$)

I. Expectation of N(t)

$$(3.14)\qquad E[N(t)] = E[N(0) + M(t) + S(t) - R(t)]$$

$$= E[M(t)] + E[S(t)] - E[R(t)].$$

From Table 3.1, we obtain

$$E[N(t)] = \begin{cases} [M\lambda + (S - n)\alpha + n\mu]t, & \text{for } 0 \le n < R, \\ [M\lambda + (S - n)\alpha + R\mu]t, & \text{for } R \le n \le S, \\ [(N - n)\lambda + R\mu]t, & \text{for } S < n \le N. \end{cases}$$

II. Variance of N(t)

$$(3.15)\qquad \mathrm{Var}[N(t)] = \mathrm{Var}[N(0) + M(t) + S(t) - R(t)]$$

$$= \mathrm{Var}[M(t)] + \mathrm{Var}[S(t)] + \mathrm{Var}[R(t)]$$

$$+ 2\,\mathrm{Cov}[M(t), S(t)] - 2\,\mathrm{Cov}[M(t), R(t)]$$

$$- 2\,\mathrm{Cov}\,[S(t), R(t)].$$

Since under heavy traffic condition, $M(t)$, $S(t)$, and $R(t)$ are approximately independent, then their covariance terms can be neglected, and we can write

$$(3.16)\qquad \mathrm{Var}[N(t)] \approx \mathrm{Var}[N(0) + M(t) + S(t) - R(t)].$$

From Table 3.1 again, we obtain

$$\mathrm{Var}[N(t)] = [M\lambda^3\sigma_M^2 + (S - n)\alpha^3\sigma_S^2 + n\mu^3\sigma_R^2]t, \qquad \text{for } 0 \le n < R,$$

$$\mathrm{Var}[N(t)] = [M\lambda^3\sigma_M^2 + (S - n)\alpha^3\sigma_S^2 + R\mu^3\sigma_R^2]t, \qquad \text{for } R \le n \le S,$$

$$\mathrm{Var}[N(t)] = [(N - n)\lambda^3\sigma_M^2 + R\mu^3\sigma_R^2]t, \qquad \text{for } S < n \le N.$$

To establish that $\{N(t), t \ge 0\}$ is approximately a diffusion process, we have to show that (i) it is stationary; and (ii) it has independent increments.

Since $E[N(t)]$ and $\mathrm{Var}[N(t)]$ are both linear functions of t, the process $\{N(t), t \ge 0\}$ is stationary. Although we can show that $\{N(t), t \ge 0\}$ is stationary, nevertheless, demonstrating that the process has independent increments is very difficult. There is no evidence that under asymptotic conditions a renewal counting process tends to normality with the additional stipulation that a renewal process with independent increments is generated. To overcome this difficulty, a heuristic argument is presented which relies on the following observation made by a number of previous authors.

In the M/M/R system, it can be shown that if the number of failed machines in the system at time t, N(t) is replaced by X(t), the birth and death equations for P(n;t), the probability mass function of N(t), i.e. the probability of n failed machines in the system at time t reduces to a Fokker-Planck equation for f(x;t) which approximates P(n;t) and which is given by equation 1.2. It is then argued that the infinitesimal mean $A_1(x)$ and variance $A_2(x)$ which appear in equation 1.2 for the M/M/R system can be modified to reflect the infinitesimal mean and variance of a G/G/R system without affecting the general structure of the Fokker-Planck equation. The new infinitesimal mean and variance are established using the renewal theoretic arguments and are used in the Fokker-Planck equation to approximate the probability density function for the queue length in the G/G/R system. This suggests that in equation 1.7, the following diffusion parameters shown in Table 3.7 for $R \le S$, and Table 3.8 for $R > S$ be used.

Table 3.7 The diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ when $R \le S$.

i	Range of x	$A_{1i}(x)$	$A_{2i}(x)$
1	$0 \le x < R$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda C_M + (S - x)\alpha C_S + x\mu C_R$
2	$R \le x \le S$	$M\lambda + (S - x)\alpha - R\mu$	$M\lambda C_M + (S - x)\alpha C_S + R\mu C_R$
3	$S < x \le N$	$(N - x)\lambda - R\mu$	$(N - x)\lambda C_M + R\mu C_R$

Table 3.8 The diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ when $R > S$.

i	Range of x	$A_{1i}(x)$	$A_{2i}(x)$
4	$0 \leq x \leq S$	$M\lambda + (S - x)\alpha - x\mu$	$M\lambda C_M + (S - x)\alpha C_S + x\mu C_R$
5	$S < x \leq R$	$(N - x)\lambda - x\mu$	$(N - x)\lambda C_M + x\mu C_R$
6	$R < x \leq N$	$(N - x)\lambda - R\mu$	$(N - x)\lambda C_M + R\mu C_R$

3.1.3 Discussions

We now discuss the operational validity of the assumptions and the reasonable applicability of a diffusion. In what follows, these two points are taken up separately.

I. Operational Validity of the Assumptions

The use of diffusion approximation is an alternative avenue to provide approximate closed form solutions to the G/G/R machine repair problem. The use of this approximation relies on the assumption that the departure process is independent of the arrival process. That is however one of the difficulties encountered by almost all diffusion approximation problems since:

1. We all know that the departure process depends of the arrival process.
2. Theoretical justification is almost impossible to provide.

Heuristically, it is intuitive that if the number of customers in the system is not zero, the independence criteria are justified to a certain extent. Ultimately however, the justification in making use of the assumptions underlying the diffusion approximation methodology and establishing operational validity, lies in the end results. Specifically, one should look on how well the numerical results obtained

using the approximation on certain prototype problems agree with the numerical results for the exact solutions to these problems. The examples that we have provided in Section 5.2.3 (Chapter 5) show that despite the restrictive independence assumptions made, the results obtained for the steady-state expected queue length of failed machines in the repair stage, are in very close agreement to the exact solution results which are available in the present literature. Additionally, it should be noted from Table 3.13 that for large λ/μ, M and S, the probability of an empty system is almost always zero, thus justifying at least for the examples work out, the independence assumption.

II. Reasonable Applicability of a Diffusion

The idea is to show that the process $\{N(t), t \geq 0\}$ which represents in our case the total number of failed machines (items) in the queue (waiting plus being serviced) can be approximated by a diffusion process. Here, there are two distinct approaches to this problem.

The first one consists in showing that $\{N(t), t \geq 0\}$ is a Gaussian process which is stationary and which has independent increments such that for $0 < t_1 < t_2 < t_3 < \ldots$, $N(t_1) - N(0)$, $N(t_2) - N(t_1)$, $N(t_3) - N(t_2)$, ..., are independently distributed with means and variances proportional to the time differences $(t_1 - 0)$, $(t_2 - t_1)$, $(t_3 - t_2)$, ...

The second one consists in showing that the continuous approximation process $\{X(t), t \geq 0\}$ of the discrete process $\{N(t), t \geq 0\}$ satisfies the Fokker-Planck equation for $f(x;t)$, the probability density function of $X(t)$.

Although one can show that $\{N(t),\ t \geq 0\}$ is stationary, nevertheless, demonstrating that the process has independent increments is very difficult. This is due to the fact that $N(t)$'s normal approximation is a result of the asymptotic normality property of the superposition of renewal processes. There is no evidence that under asymptotic conditions a renewal counting process tends to normality with the additional stipulation that a normal process with independent increments is generated. In fact, establishing such property necessitates the derivation of the probability law of a renewal counting process, i.e., the joint distribution function of the number of counts at distinct time epochs. Unfortunately, this problem is yet unsolved.

To overcome these difficulties, a heuristic argument is presented which relies on the following observation made by a number of previous authors:

In the M/M/R system, it can be shown that if $N(t)$ is replaced by $X(t)$, the birth and death equations for $P(n;t)$, the probability of n failed machines in the system at time t, reduces to a Fokker-Planck equation for $f(x;t)$ which approximates $P(n;t)$. It is then argued that the infinitesimal means and variances which appear in this equation can be modified to reflect the infinitesimal means and variances of a G/G/R system without affecting the general structure of the Fokker-Planck equation. These new infinitesimal means and variances are established using the previously mentioned renewal theoretic arguments to show that $\{N(t),\ t \geq 0\}$ is approximately a normal process.

3.2 Steady-State Diffusion Equations and Solutions to the G/G/R Model

The probability density function (p.d.f.) $f(x;t)$ is divided into three different ranges of x for the case when $R \leq S$ and the case when $R > S$, respectively:

(i) for $R \leq S$

$$f(x;t) = \begin{cases} f_7(x;t), & \text{for } 0 \leq x < R, \\ f_8(x;t), & \text{for } R \leq x \leq S, \\ f_9(x;t), & \text{for } S < x \leq N, \end{cases} \tag{3.17}$$

and (ii) for $R > S$

$$f(x;t) = \begin{cases} f_{10}(x;t), & \text{for } 0 \leq x \leq S, \\ f_{11}(x;t), & \text{for } S < x \leq R, \\ f_{12}(x;t), & \text{for } R < x \leq N. \end{cases} \tag{3.18}$$

Based on our previous arguments, the p.d.f. $f(x;t)$ satisfies the Fokker-Planck equation with diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$, for $i = 1, 2, \ldots, 6$. The p.d.f $f(x;t)$ is a solution to the Fokker-Planck equation

$$\frac{\partial f_j(x;t)}{\partial t} = -\frac{\partial}{\partial x}[A_{1i}(x)f_j(x;t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_{2i}(x)f_j(x;t)], \tag{3.19}$$

where $i = 1, 2, \ldots, 6$, and $j = 7, 8, \ldots, 12$.

Under steady-state conditions, when $\lim_{t \to \infty} f_j(x;t) = f_j(x)$, the Fokker-Planck equation 3.19 takes the form

$$-\frac{d}{dx}[A_{1i}(x)f_j(x)] + \frac{1}{2}\frac{d^2}{dx^2}[A_{2i}(x)f_j(x)] = 0. \tag{3.20}$$

3.2.1 Steady-State Diffusion Equations for the G/G/R Model

In the steady-state, substituting the values of $A_{1i}(x)$ and $A_{2i}(x)$ given by Table 3.7 in equation 3.20, we obtain for $R \le S$ the following steady-state equations for the p.d.f. $f(x)$, where x is the number of failed machines in the G/G/R MRP with spares.

$$(3.21a) \qquad 0 = -\frac{d}{dx}\{[M\lambda + (S - x)\alpha - x\mu]\, f_7(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda C_M + (S - x)\alpha C_S + x\mu C_R}{2}\, f_7(x)\}, \qquad 0 \le x < R,$$

$$(3.21b) \qquad 0 = -\frac{d}{dx}\{[M\lambda + (S - x)\alpha - R\mu]\, f_8(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda C_M + (S - x)\alpha C_S + R\mu C_R}{2}\, f_8(x)\}, \qquad R \le x \le S,$$

$$(3.21c) \qquad 0 = -\frac{d}{dx}\{[(N - x)\lambda - R\mu]\, f_9(x)\} + \frac{d^2}{dx^2}\{\frac{(N - x)\lambda C_M + R\mu C_R}{2}\, f_9(x)\}, \qquad S < x \le N.$$

Likewise for $R > S$, using the results of Table 3.8 in equation 3.20, we obtain the following steady-state equations for the p.d.f. $f(x)$ in the G/G/R MRP with spares.

$$(3.22a) \qquad 0 = -\frac{d}{dx}\{[M\lambda + (S - x)\alpha - x\mu]\, f_{10}(x)\} + \frac{d^2}{dx^2}\{\frac{M\lambda C_M + (S - x)\alpha C_S + x\mu C_R}{2}\, f_{10}(x)\}, \qquad 0 \le x \le S,$$

$$(3.22b) \qquad 0 = -\frac{d}{dx}\{[(N-x)\lambda - x\mu]\, f_{11}(x)\} + \frac{d^2}{dx^2}\{\frac{(N-x)\lambda C_M + x\mu C_R}{2}\, f_{11}(x)\}, \quad S < x \le R,$$

$$(3.22c) \qquad 0 = -\frac{d}{dx}\{[(N-x)\lambda - R\mu]\, f_{12}(x)\} + \frac{d^2}{dx^2}\{\frac{(N-x)\lambda C_M + R\mu C_R}{2}\, f_{12}(x)\}, \quad R < x \le N.$$

3.2.2 Approximate Formulas for the Probability Density Function f(x) to the G/G/R Model

The steady-state solutions of equations 3.21 subject to the boundary condition specified by equation 1.13 are given by

$$(3.23) \qquad f_j(x) = \frac{K_j}{A_{2i}(x)} \exp\{\int^x \frac{2A_{1i}(y)}{A_{2i}(y)}\, dy\}, \quad \text{for} \quad 0 \le x \le N, \text{ and}$$

$i = 1, 2, \ldots, 6$, where $K_j > 0$ ($j = 7, 8, \ldots, 12$) is the constant of integration.

Using the results of Table 3.7 in equation 3.23, we obtain for $R \le S$ the following expressions for $f_7(x)$, $f_8(x)$, and $f_9(x)$ in terms of the dimensionless quantities $\theta_\lambda = \lambda/\mu$ and $\theta_\alpha = \alpha/\mu$:

$$\text{(3.24a)} \qquad f_7(x) = \frac{K_7}{M\lambda C_M+(S-x)\alpha C_S+x\mu C_R} \exp\left(\int_0^x \frac{2[M\lambda+(S-y)\alpha-y\mu]}{M\lambda C_M+(S-y)\alpha C_S+y\mu C_R} dy \right)$$

$$= K_7[M\theta_\lambda C_M+(S-x)\theta_\lambda C_S+x\mu C_R]^{-1} \left[\frac{M\theta_\lambda C_M+(S-x)\theta_\alpha C_S+xC_R}{M\theta_\lambda C_M+S\theta_\alpha C_S}\right]^{\beta_7}$$

$$* \exp\left(\frac{-2(\theta_\alpha+1)x}{C_R - \theta_\alpha C_S}\right) \qquad \text{if } C_R - \theta_\alpha C_S \neq 0$$

$$= K_7(M\theta_\lambda C_M + S\theta_\lambda C_S)^{-1} \exp\left(\frac{2(M\theta_\lambda+S\theta_\alpha)x}{M\theta_\lambda C_M+S\theta_\alpha C_S} - \frac{(\theta_\alpha+1)x^2}{M\theta_\lambda C_R + S\theta_\alpha C_S}\right) \qquad \text{if } C_R - \theta_\alpha C_S = 0$$

$$= K_7 g_7(x), \quad \text{for } 0 \leq x < R,$$

$$\text{(3.24b)} \qquad f_8(x) = \frac{K_8}{M\lambda C_M+(S-x)\alpha C_S+R\mu C_R} \exp\left(\int_R^x \frac{2[M\lambda+(S-y)\alpha-R\mu]}{M\lambda C_M+(S-y)\alpha C_S+R\mu C_R} dy \right)$$

$$= K_8[M\theta_\lambda C_M+(S-x)\theta_\alpha C_S+RC_R]^{-1} \left[\frac{M\theta_\lambda C_M+(S-x)\theta_\alpha C_S+RC_R}{M\theta\lambda CM+(S-R)\theta\alpha CS+RCR}\right]^{\beta_8}$$

$$* \exp\left(\frac{2(x-R)}{C_S}\right) \qquad \text{if } C_S > 0$$

$$= K_8(M\theta_\lambda C_M + RC_R)^{-1} \exp\left(\frac{2(M\theta_\lambda+S\theta_\alpha-R)(x-R)}{M\theta_\lambda C_M + RC_R} - \frac{\theta_\alpha(x^2 - R^2)}{M\theta_\lambda C_R + RC_R}\right) \qquad \text{if } C_S = 0$$

$$= K_8 g_8(x), \quad \text{for } R \leq x \leq S$$

$$(3.24c) \qquad f_9(x) = \frac{K_9}{(N-x)\lambda C_M + R\mu C_R} \exp\left(\int_S^x \frac{2[(N-y)\lambda - R\mu]}{(N-y)\lambda C_M + R\mu C_R} dy \right)$$

$$= K_9[(N-x)\theta_\lambda C_M + RC_R]^{-1} \left[\frac{(N-x)\theta_\lambda C_M + RC_R}{M\theta_\lambda C_M + RC_R} \right]^{\beta_9}$$

$$* \exp\left(\frac{2(x-S)}{C_M} \right) \qquad \text{if } C_M > 0$$

$$= K_9(RC_R)^{-1} \exp\left(\frac{2(N\theta_\lambda - R)(x-S)}{RC_R} - \frac{\theta_\lambda(x^2-S^2)}{RC_R} \right) \text{ if } C_M = 0$$

$$= K_9 g_9(x), \quad \text{for } S < x \le N$$

where

$$\beta_7 = \frac{2M\theta_\lambda\left[1 + \frac{(\theta_\alpha+1)C_M}{C_R - \theta_\alpha C_S}\right] + 2S\theta_\alpha\left[1 + \frac{(\theta_\alpha+1)C_S}{C_R - \theta_\alpha C_S}\right]}{C_R - \theta_\alpha C_S},$$

$$\beta_8 = \frac{2M\theta_\lambda\left(1 - \frac{C_M}{C_S}\right) - 2R\left(1 + \frac{C_R}{C_S}\right)}{-\theta_\alpha C_S},$$

$$\beta_9 = \frac{2R\left(1 + \frac{C_R}{C_M}\right)}{\theta_\lambda C_M}.$$

Likewise for $R > S$, using the results of Table 3.8 in equation 3.23, we obtain the following expressions for $f_7(x)$, $f_8(x)$, and $f_9(x)$ in terms of the dimensionless quantities $\theta_\lambda = \lambda/\mu$ and $\theta_\alpha = \alpha/\mu$:

$$(3.25a)\qquad f_{10}(x) = \frac{K_{10}}{M\lambda C_M+(S-x)\alpha C_S+x\mu C_R} \exp\{\int_0^x \frac{2[M\lambda+(S-y)\alpha-y\mu]}{M\lambda C_M+(S-y)\alpha C_S+y\mu C_R} dy\}$$

$$= K_{10}[M\theta_\alpha C_M+(S-x)\theta_\alpha C_S+xC_R]^{-1}[\frac{M\theta_\lambda C_M+(S-x)\theta_\alpha C_S+xC_R}{M\theta_\lambda C_M+S\theta_\alpha C_S}]^{\beta_{10}}$$

$$* \exp\{\frac{-2(\theta_\alpha+1)x}{C_R - \theta_\alpha C_S}\} \qquad \text{if } C_R - \theta_\alpha C_S \neq 0$$

$$= K_{10}(M\theta_\lambda C_M + S\theta_\alpha C_S)^{-1} \exp\{\frac{2(M\theta_\lambda+S\theta_\alpha)x}{M\theta_\lambda C_M+S\theta_\alpha C_S} - \frac{(\theta_\alpha+1)x^2}{M\theta_\lambda C_R + S\theta_\alpha C_S}\} \qquad \text{if } C_R - \theta_\alpha C_S = 0$$

$$= K_{10}g_{10}(x), \quad \text{for } 0 \le x \le S$$

$$(3.25b)\qquad f_{11}(x) = \frac{K_{11}}{(N-x)\lambda C_M+x\mu C_R} \exp\{\int_S^x \frac{2[(N-y)\lambda-y\mu]}{(N-y)\lambda C_M+y\mu C_R} dy\}$$

$$= K_{11}[(N-x)\theta_\lambda C_M+xC_R]^{-1} [\frac{(N-x)\theta_\lambda C_M+xC_R}{M\theta_\lambda C_M+SC_R}]^{\beta_{11}}$$

$$* \exp\{\frac{2(\theta_\lambda+1)(x-S)}{\theta_\lambda C_M - C_R}\} \qquad \text{if } \theta_\lambda C_M-C_R \neq 0$$

$$= K_{11}(N\theta_\lambda C_M)^{-1}\exp\{\frac{2(x-S)}{C_M} - \frac{(\theta_\lambda+1)(x^2-S^2)}{N\theta_\lambda C_M}\} \text{ if } \theta_\lambda C_M-C_R = 0$$

$$= K_{11}g_{11}(x), \quad \text{for } S < x \le R$$

(3.25c)
$$f_{12}(x) = \frac{K_{12}}{(N-x)\lambda C_M + R\mu C_R} \exp\{\int_R^x \frac{2[(N-y)\lambda - R\mu]}{(N-y)\lambda C_M + R\mu C_R}\, dy\}$$

$$= K_{12}[(N-x)\theta_\lambda C_M + RC_R]^{-1} [\frac{(N-x)\theta_\lambda C_M + RC_R}{(N-R)\theta_\lambda C_M + RC_R}]^{\beta_{12}}$$

$$* \exp\{\frac{2(x-R)}{C_M}\} \qquad \text{if } C_M > 0$$

$$= K_{12}(RC_R)^{-1} \exp\{\frac{2(N\theta_\lambda - R)(x-R)}{RC_R} - \frac{\theta_\lambda(x^2-R^2)}{RC_R}\} \text{ if } C_M = 0$$

$$\equiv K_{12}g_{12}(x), \quad \text{for } R < x \leq N$$

where

$$\beta_{10} = \frac{2M\theta_\lambda[1 + \frac{(\theta_\alpha+1)C_M}{C_R - \theta_\alpha C_S}] + 2S\theta_\alpha[1 + \frac{(\theta_\alpha+1)C_S}{C_R - \theta_\alpha C_S}]}{C_R - \theta_\alpha C_S},$$

$$\beta_{11} = \frac{2N\theta_\lambda[1 + \frac{(\theta_\lambda+1)C_M}{C_R - \theta_\lambda C_M}]}{C_R - \theta_\lambda C_M},$$

$$\beta_{12} = \frac{2R(1 + \frac{C_R}{C_M})}{\theta_\lambda C_M}.$$

Solutions $f_7(x)$, $f_8(x)$, and $f_9(x)$ can be verified to satisfy the boundary condition specified by equation 1.13. Equations 3.24a through 3.24c can be rewritten as

(3.26a) $\quad f_7(x) = K_7 g_7(x), \quad$ for $\quad 0 \leq x < R,$

(3.26b) $\quad f_8(x) = K_8 g_8(x), \quad$ for $\quad R \leq x \leq S,$

(3.26c) $\quad f_9(x) = K_9 g_9(x), \quad$ for $\quad S < x \leq N.$

Similar type expressions for $R > S$ can be written for equations 3.25a

through 3.25c.

3.2.3 Determination of f(x), P_0, E[X], and Var[X]

For R ≤ S, the unknown constants K_7, K_8, and K_9 are determined by two conditions:

The first condition comes from the normalization criterion, namely,

$$(3.27) \qquad 1 = \int_0^N f(x)dx = K_7 \int_0^R g_7(x)dx + K_8 \int_R^S g_8(x)dx + K_9 \int_S^N g_9(x)dx.$$

The second condition requires continuity of f(x) at x = R and x = S; i.e., $f_7(R) = f_8(R)$, and $f_8(S) = f_9(S)$. Therefore we obtain

$$(3.28a) \qquad K_7 g_7(R) = K_8 g_8(R),$$

and

$$(3.28b) \qquad K_8 g_8(S) = K_9 g_9(S).$$

The unknown constants K_7, K_8, and K_9 are determined by solving equations 3.27 and 3.28 simultaneously. Likewise for R > S, the unknown constants K_{10}, K_{11}, and K_{12} are obtained from the first and the second conditions. Once the unknown constants K_7, K_8, and K_9 are obtained, f(x) can be determined for the three different ranges of x. This will specify the p.d.f. f(x) of X, the total number of failed machines in the system in steady-state.

The steady-state probabilities P_n, n ≥ 0, of n failed machines in the system can be obtained by discretizing f(x) using for example the procedures suggested by Halachmi and Franta [25]

$$(3.29) \qquad P_n = \int_{n-0.5}^{n+0.5} f(x)dx, \qquad n = 1, 2, \ldots, N-1,$$

$$(3.30) \qquad P_0 = \int_{0.5}^{0.5} f(x)dx,$$

and

$$(3.31) \qquad P_N = \int_{N-0.5}^{N} f(x)dx.$$

The mean E[X], and the variance Var[X] for the G/G/R MRP are given by

$$(3.32) \qquad E[X] = K_7 \int_0^R xg_7(x)dx + K_8 \int_R^S xg_8(x)dx + K_9 \int_S^N xg_9(x)dx,$$

and

$$(3.33) \qquad Var[X] = E[X^2] - (E[X])^2,$$

where

$$E[X^2] = K_7 \int_0^R x^2g_7(x)dx + K_8 \int_R^S x^2g_8(x)dx + K9 \int_S^N x^2g_9(x)dx.$$

Equivalent results may be obtained when $R > S$. Thus E[X] and Var[X] would provide approximate results respectively to

$$E[N] = \sum_{n=0}^{N} nP_n \quad \text{and} \quad Var[N] = \sum_{n=0}^{N} \{n - E[N]\}^2P_n,$$

where the steady-state solutions Pn are given in equations 2.3 for $R \leq S$ and given in equations 2.4 for $R > S$.

3.3 Computational Techniques and Output

It is very difficult to determine analytically the constants K_7, K_8, and K_9. Since the functions $g_7(x)$, $g_8(x)$, and $g_9(x)$ cannot be integrated analytically, we will use numerical techniques to integrate these functions. Romberg integration is a method which uses the trapezoidal rule to give preliminary approximations, prior to applying the Richardson extrapolation technique to improve the approximations. This method will be used to estimate the following integrals:

$$\text{(a)} \int_0^R g_7(x)dx, \text{ (b)} \int_R^S g_8(x)dx, \text{ and (c)} \int_S^N g_9(x)dx.$$

It is easy to compute the values of the functions $g_7(x)$, $g_8(x)$, and $g_9(x)$ at $x = R$ and $x = S$, respectively, and then substitute these values into equation 3.27. Therefore from equations 3.27 and 3.28, we obtain the following three equations:

$$\text{(3.34a)} \quad K_7 \int_0^R g_7(x)dx + K_8 \int_R^S g_8(x)dx + K_9 \int_S^N g_9(x)dx = 1,$$

$$\text{(3.34b)} \quad K_7 g_7(R) + K_8(-g_8(R)) = 0,$$

$$\text{(3.34c)} \quad K_8 g_8(S) + K_9(-g_9(S)) = 0.$$

Similar type equations may be obtained when $R > S$.

We use the Gaussian elimination with backward substitution algorithm to solve the linear equations 3.34 for the unknown constants K_7, K_8, and K_9. Likewise, following the same procedure, we can obtain the unknown constants K_{10}, K_{11}, and K_{12} in equations 3.25a through 3.25c.

3.3.1 Numerical Results, Justification of the Diffusion Approximation

We present the approximate results for the mean and variance of the number of failed machines for the M/M/R, M/G/R, G/M/R and G/G/R machine repair models with warm standbys.

The examples considered assume the following:

(a) M/M/R: $C_M = 1.0$, $C_S = 1.0$, and $C_R = 1.0$,

(b) M/G/R: $C_M = 1.0$, $C_S = 1.0$, and $C_R = 0.25$,

(c) G/M/R: $C_M = 0.25$, $C_S = 0.04$. and $C_R = 1.0$,

(d) G/G/R: $C_M = 0.25$, $C_S = 0.04$, and $C_R = 0.25$.

The numerical results of the E[X] and Var[X] are shown in Tables 3.9 through 3.12 for the range $0.2 \le \theta_\lambda \le 2.0$ in increments of 0.2, and for $\theta_\alpha = 0.05$. In these numerical examples, there are M = 10 (operating machines), S = 5 (spares), and various number of repairmen.

One sees from Tables 3.9 through 3.12 that the square coefficients of variation C_M, C_S and C_R do not significantly affect E[X]. However, differences in Var[X] can be noted when C_M, C_S and C_R are varied; lower coefficient of variation leads to lower Var[X].

In Chapter 4, diffusion approximation will provide a useful analytic tool to obtain system characteristics of complex machine repair problems and thus is helpful in analyzing real life situations which otherwise would have been too complex to solve analytically.

Additional numerical results for the G/G/R MRP with warm standbys are presented in Table 3.13. Note that in Table 3.13, $1 - P_0$ is very close to 1.0, thus justifying the assumption of heavy traffic approximation. The numerical results shown in Table 3.13 show that the probability of a non-empty repair stage $1 - P_0$ is very close to unity for moderate to large values of (1) $\theta_\lambda = \lambda/\mu$ and (2) M and S.

3.3.2 Plot of the f(x) for the M/M/R, M/G/R, G/M/R, and G/G/R Models

After obtaining the values of K_7, K_8, and K_9, we have plotted the steady-state probabilities f(x) for three different ranges of x; i.e., $K_7 g_7(x)$, for $0 \le x \le R$; $K_8 g_8(x)$ for $R \le x \le S$; and $K_9 g_9(x)$, for $S \le x \le N$ for the M/M/R, M/G/R, G/M/R, and G/G/R models. The steady-state probabilities f(x) for the M/M/R, M/G/R, G/M/R, and G/G/R machine repair models are plotted in Figures 3.1 through 3.9 for $\theta_\alpha = 0.05$, and $\theta_\lambda = 0.2$, 0.4, and 0.6, and for various values of R (R = 1, 3, 5).

The figures depicted come in many different shapes. Some shapes occur with enough regularity that they have been given special names. For examples, the probability that few machines are in a failed state is low while the probability that a large number of machines are in a failed state is high, the longer tail of the density function f(x) extends toward the origin. The probability that a large number of machines are in a failed state is low while the probability that only a few machines are in a failed state is high, the longer tail of the density function f(x) extends away from the origin.

From Figures 3.1 through 3.9, we observe that the means of the M/M/R, M/G/R, G/M/R, and G/G/R models nearly determine the locations of the center of the curve, and the standard deviations determine its flatness. The graph with the smaller standard deviation is less spread out and more sharply peaked. From Tables 3.9 through 3.12, the variances of the G/G/R model are smaller than those of the M/G/R (or G/M/R) model, and the variances of the M/G/R (or G/M/R) model are smaller than those of the M/M/R model. For example, the curves of the G/G/R model (R = 1, 3, 5) with smaller standard deviations are more peaked than the M/G/R (or G/M/R) model with greater standard deviations, for the values of $\theta_\lambda = 0.2$, 0.4 and 0.6, and the curves of the M/G/R (or G/M/R) model with smaller standard deviations are more peaked than the M/M/R model with greater standard deviations, for the values of $\theta_\lambda = 0.2$, 0.4 and 0.6 (see Figures 3.1 through 3.9). In general, it is hard to compare the flatness of the curves of the M/G/R model with those of the G/M/R model for various values of R, and for $0.2 \leq \theta_\lambda \leq 2.0$, and for $\theta_\alpha = 0.05$.

Let us describe two important characteristics of f(x) as follows. First, fix the values of θ_λ, the modes of the curves of the M/M/R, M/G/R, G/M/R, and G/G/R models with larger number of repairmen are to the left of those models with lower number of repairmen. Second, fix the number of repairmen, the modes of the curves of the M/M/R, M/G/R, G/M/R, and G/G/R models with larger values θ_λ are to the right of those models with smaller value θ_λ.

Note that in Theorem 1.1 (Chapter 1), the assumption of tail conditions for f(x), namely, at least at one of the tails of f(x) defined in the interval $0 \leq x \leq N$ vanishes and its first derivative also vanishes is made. It is of interest to note that from Figures 3.1 through 3.9, at least one of the tails of the curve f(x) defined in the interval $0 \leq x \leq N$ vanish for the M/M/R, M/G/R, G/M/R, and G/G/R models; i.e., f(x) and its first derivative vanish at $x = 0$, or vanish at $x = N$, or vanish at $x = 0$ and $x = N$. Therefore, the justification and validation in making use of the assumption of tail conditions for f(x) in Theorem 1.1 is established empirically from the results obtained so far.

Table 3.9 The approximate mean and variance for the M/M/R MRP with warm standbys ($C_M = 1.0$, $C_S = 1.0$, $C_R = 1.0$, $M = 10$, $S = 5$, $N = 15$, $\theta_\alpha = 0.05$).

θ_λ	R :	1	2	3	4	5
0.2	E[X]	9.9620	5.0548	2.9240	2.4144	2.2887
	Var[X]	5.0417	7.8370	4.0021	2.4616	2.0068
0.4	E[X]	12.4062	9.9591	7.4099	5.4579	4.5093
	Var[X]	2.3054	5.0584	7.4870	6.7872	5.0106
0.6	E[X]	13.1823	11.6093	9.9620	8.3056	6.9336
	Var[X]	1.4677	3.1835	5.0318	6.5556	6.6006
0.8	E[X]	13.5626	12.4062	11.2041	9.9694	8.7703
	Var[X]	1.0743	2.3056	3.6323	4.9740	5.9309
1.0	E[X]	13.7895	12.8745	11.9304	10.9601	9.9827
	Var[X]	0.8504	1.7964	2.8282	3.8945	4.8826
1.2	E[X]	13.9410	13.1823	12.4062	11.6096	10.7999
	Var[X]	0.7072	1.4677	2.3055	3.1799	4.0459
1.4	E[X]	14.0500	13.4001	12.7415	12.0671	11.3800
	Var[X]	0.6082	1.2402	1.9401	2.6769	3.4242
1.6	E[X]	14.1326	13.5626	12.9903	12.4062	11.8109
	Var[X]	0.5359	1.0743	1.6719	2.3052	2.9554
1.8	E[X]	14.1975	13.6887	13.1823	12.6673	12.1429
	Var[X]	0.4809	0.9487	1.4677	2.0205	2.5924
2.0	E[X]	14.2500	13.7895	13.3349	12.8745	12.4063
	Var[X]	0.4375	0.8504	1.3077	1.7963	2.3046

Table 3.9--Continued

θ_λ	R :	6	7	8	9	10
0.2	E[X]	2.2572	2.2497	2.2480	2.2477	2.2476
	Var[X]	1.8726	1.8350	1.8254	1.8232	1.8228
0.4	E[X]	4.1454	4.0200	3.9807	3.9697	3.9670
	Var[X]	3.9704	3.5189	3.3489	3.2926	3.2763
0.6	E[X]	6.0895	5.6805	5.5123	5.4521	5.4333
	Var[X]	5.6253	4.7396	4.2369	4.0107	3.9255
0.8	E[X]	7.7817	7.1349	6.7938	6.6434	6.5871
	Var[X]	5.9654	5.3493	4.7083	4.3025	4.1067
1.0	E[X]	9.0685	8.3388	7.8649	7.6138	7.5035
	Var[X]	5.4554	5.3749	4.9025	4.4372	4.1450
1.2	E[X]	10.0045	9.2943	8.7568	8.4237	8.2551
	Var[X]	4.7512	5.0394	4.8582	4.4692	4.1362
1.4	E[X]	10.6918	10.0398	9.4924	9.1079	8.8876
	Var[X]	4.1123	4.5707	4.6423	4.4032	4.0911
1.6	E[X]	11.2100	10.6228	10.0956	9.6878	9.4282
	Var[X]	3.5866	4.0980	4.3354	4.2551	4.0078
1.8	E[X]	11.6118	11.0844	10.5903	10.1797	9.8942
	Var[X]	3.1624	3.6716	3.9997	4.0518	3.8891
2.0	E[X]	11.9315	11.4559	10.9983	10.5976	10.2981
	Var[X]	2.8187	3.3031	3.6718	3.8202	3.7432

Table 3.10 The approximate mean and variance for the M/G/R MRP with warm standbys ($C_M = 1.0$, $C_S = 1.0$, $C_R = 0.25$, $M = 10$, $S = 5$, $N = 15$, $\theta_\alpha = 0.05$).

θ_λ	R :	1	2	3	4	5
0.2	E[X]	9.9934	4.8029	2.5726	2.2485	2.2079
	Var[X]	3.1679	5.5412	2.0980	1.2812	1.1597
0.4	E[X]	12.4968	9.9924	7.4119	5.2689	4.3212
	Var[X]	1.5565	3.1739	4.9279	4.4901	2.9881
0.6	E[X]	13.3195	11.6657	9.9931	8.3121	6.8420
	Var[X]	1.0254	2.0836	3.1674	4.2116	4.3046
0.8	E[X]	13.7221	12.4968	11.2490	9.9954	8.7621
	Var[X]	0.7595	1.5565	2.3479	3.1501	3.8101
1.0	E[X]	13.9581	12.9923	11.9986	10.9989	10.0004
	Var[X]	0.6023	1.2384	1.8728	2.5055	3.1178
1.2	E[X]	14.1123	13.3195	12.4968	11.6658	10.8330
	Var[X]	0.4998	1.0254	1.5565	2.0831	2.6042
1.4	E[X]	14.2208	13.5507	12.8512	12.1412	11.4280
	Var[X]	0.4283	0.8732	1.3295	1.7824	2.2315
1.6	E[X]	14.3011	13.7221	13.1154	12.4968	11.8740
	Var[X]	0.3759	0.7595	1.1586	1.5564	1.9509
1.8	E[X]	14.3629	13.8538	13.3195	12.7726	12.2203
	Var[X]	0.3361	0.6718	1.0254	1.3800	1.7320
2.0	E[X]	14.4121	13.9581	13.4816	12.9923	12.4969
	Var[X]	0.3048	0.6023	0.9188	1.2384	1.5563

Table 3.10--Continued

θ_λ	R :	6	7	8	9	10
0.2	E[X]	2.2039	2.2037	2.2037	2.2037	2.2037
	Var[X]	1.1453	1.1443	1.1442	1.1442	1.1442
0.4	E[X]	4.0543	3.9985	3.9904	3.9896	3.9896
	Var[X]	2.3421	2.1715	2.1410	2.1376	2.1374
0.6	E[X]	5.9574	5.6028	5.5026	5.4831	5.4807
	Var[X]	3.4605	2.8058	2.5491	2.4853	2.4757
0.8	E[X]	7.7017	7.0242	6.7223	6.6285	6.6091
	Var[X]	3.8330	3.2766	2.7799	2.5579	2.4978
1.0	E[X]	9.0429	8.2548	7.7662	7.5523	7.4883
	Var[X]	3.5217	3.4247	2.9863	2.6303	2.4781
1.2	E[X]	10.0102	9.2516	8.6708	8.3406	8.2099
	Var[X]	3.0626	3.2620	3.0670	2.7108	2.4745
1.4	E[X]	10.7170	10.0299	9.4358	9.0276	8.8258
	Var[X]	2.6595	2.9705	2.9954	2.7479	2.4799
1.6	E[X]	11.2506	10.6363	10.0687	9.6246	9.3631
	Var[X]	2.3365	2.6696	2.8270	2.7178	2.4759
1.8	E[X]	11.6663	11.1156	10.5882	10.1387	9.8358
	Var[X]	2.0791	2.4013	2.6218	2.6293	2.4497
2.0	E[X]	11.9989	11.5016	11.0161	10.5788	10.2524
	Var[X]	1.8709	2.1727	2.4160	2.5044	2.3983

Table 3.11 The approximate mean and variance for the G/M/R MRP with warm standbys ($C_M = 0.25$, $C_S = 0.04$, $C_R = 1.0$, $M = 10$, $S = 5$, $N = 15$, $\theta_\alpha = 0.05$).

θ_λ	R :	1	2	3	4	5
0.2	E[X]	9.9910	4.7039	2.5692	2.2258	2.1610
	Var[X]	3.1044	5.9577	2.5000	1.5356	1.3147
0.4	E[X]	12.4499	9.9906	7.3991	5.2356	4.3326
	Var[X]	1.4410	3.1068	5.0158	4.7157	3.1965
0.6	E[X]	13.2425	11.6402	9.9907	8.3084	6.8266
	Var[X]	0.8933	1.9977	3.1061	4.2325	4.4323
0.8	E[X]	13.6310	12.4499	11.2309	9.9913	8.7548
	Var[X]	0.6332	1.4410	2.2747	3.1017	3.8372
1.0	E[X]	13.8621	12.9279	11.9657	10.9843	9.9932
	Var[X]	0.4853	1.1096	1.7752	2.4400	3.0894
1.2	E[X]	14.0161	13.2425	12.4499	11.6402	10.8197
	Var[X]	0.3915	0.8934	1.4410	1.9977	2.5489
1.4	E[X]	14.1264	13.4651	12.7921	12.1046	11.4066
	Var[X]	0.3272	0.7429	1.2037	1.6797	2.1560
1.6	E[X]	14.2098	13.6310	13.0463	12.4499	11.8439
	Var[X]	0.2807	0.6332	1.0279	1.4410	1.8587
1.8	E[X]	14.2751	13.7595	13.2425	12.7163	12.1816
	Var[X]	0.2456	0.5501	0.8934	1.2562	1.6266
2.0	E[X]	14.3279	13.8621	13.3985	12.9279	12.4499
	Var[X]	0.2183	0.4853	0.7876	1.1096	1.4410

Table 3.11--Continued

θ_λ	R :	6	7	8	9	10
0.2	E[X]	2.1482	2.1457	2.1453	2.1452	2.1452
	Var[X]	1.2616	1.2495	1.2469	1.2464	1.2463
0.4	E[X]	4.0650	3.9931	3.9753	3.9713	3.9705
	Var[X]	2.4867	2.2418	2.1675	2.1476	2.1429
0.6	E[X]	5.9777	5.6390	5.5284	5.4971	5.4893
	Var[X]	3.5823	2.9012	2.5877	2.4741	2.4396
0.8	E[X]	7.6962	7.0497	6.7632	6.6618	6.6318
	Var[X]	3.9133	3.3502	2.8356	2.5703	2.4677
1.0	E[X]	9.0315	8.2528	7.7907	7.5861	7.5141
	Var[X]	3.5633	3.4903	3.0445	2.6710	2.4816
1.2	E[X]	9.9989	9.2379	8.6729	8.3632	8.2345
	Var[X]	3.0587	3.3140	3.1260	2.7635	2.5070
1.4	E[X]	10.7029	10.0142	9.4241	9.0355	8.8457
	Var[X]	2.6208	2.9904	3.0480	2.8000	2.5243
1.6	E[X]	11.2311	10.6194	10.0511	9.6197	9.3754
	Var[X]	2.2732	2.6541	2.8603	2.7626	2.5188
1.8	E[X]	11.6402	11.0958	10.5681	10.1249	9.8392
	Var[X]	1.9974	2.3565	2.6281	2.6617	2.4843
2.0	E[X]	11.9657	11.4773	10.9939	10.5592	10.2472
	Var[X]	1.7751	2.1056	2.3945	2.5198	2.4217

Table 3.12 The approximate mean and variance for the G/G/R MRP with warm standbys ($C_M = 0.25$, $C_S = 0.04$, $C_R = 0.25$, $M = 10$, $S = 5$, $N = 15$, $\theta_\alpha = 0.05$).

θ_λ	R :	1	2	3	4	5
0.2	E[X]	10.0000	4.4499	2.2380	2.1466	2.1431
	Var[X]	1.2501	3.1439	0.6821	0.5046	0.4957
0.4	E[X]	12.4999	10.0000	7.4734	4.9882	4.1358
	Var[X]	0.6247	1.2501	1.9707	2.1405	1.1333
0.6	E[X]	13.3318	11.6667	10.0000	8.3301	6.7196
	Var[X]	0.4144	0.8333	1.2501	1.6795	1.9077
0.8	E[X]	13.7453	12.4999	11.2500	10.0000	8.7503
	Var[X]	0.3072	0.6247	0.9375	1.2501	1.5606
1.0	E[X]	13.9909	12.9994	12.0000	11.0000	10.0000
	Var[X]	0.2420	0.4989	0.7499	1.0001	1.2500
1.2	E[X]	14.1527	13.3318	12.4999	11.6667	10.8333
	Var[X]	0.1982	0.4144	0.6247	0.8333	1.0418
1.4	E[X]	14.2669	13.5685	12.8568	12.1428	11.4286
	Var[X]	0.1671	0.3534	0.5350	0.7142	0.8929
1.6	E[X]	14.3516	13.7453	13.1242	12.4999	11.8750
	Var[X]	0.1440	0.3072	0.4673	0.6247	0.7812
1.8	E[X]	14.4168	13.8821	13.3318	12.7775	12.2222
	Var[X]	0.1262	0.2710	0.4144	0.5549	0.6943
2.0	E[X]	14.4686	13.9909	13.4976	12.9994	12.4999
	Var[X]	0.1122	0.2420	0.3717	0.4989	0.6247

Table 3.12--Continued

θ_λ	R :	6	7	8	9	10
0.2	E[X]	2.1430	2.1430	2.1430	2.1430	2.1430
	Var[X]	0.4955	0.4955	0.4955	0.4955	0.4955
0.4	E[X]	4.0291	4.0212	4.0209	4.0209	4.0209
	Var[X]	0.9240	0.9031	0.9021	0.9021	0.9021
0.6	E[X]	5.8054	5.5920	5.5668	5.5653	5.5652
	Var[X]	1.3376	1.0240	0.9690	0.9646	0.9645
0.8	E[X]	7.5764	6.8773	6.6864	6.6608	6.6592
	Var[X]	1.6642	1.2613	1.0013	0.9480	0.9432
1.0	E[X]	9.0039	8.1150	7.6366	7.5149	7.5000
	Var[X]	1.4847	1.4763	1.1488	0.9746	0.9424
1.2	E[X]	10.0002	9.1816	8.5296	8.2454	8.1874
	Var[X]	1.2492	1.4098	1.2897	1.0374	0.9436
1.4	E[X]	10.7143	10.0017	9.3358	8.9073	8.7708
	Var[X]	1.0714	1.2434	1.3019	1.1105	0.9518
1.6	E[X]	11.2500	10.6252	10.0113	9.5127	9.2831
	Var[X]	0.9375	1.0929	1.2151	1.1485	0.9693
1.8	E[X]	11.6667	11.1111	10.5580	10.0522	9.7439
	Var[X]	0.8333	0.9721	1.1025	1.1321	0.9873
2.0	E[X]	12.0000	11.5000	11.0005	10.5188	10.1618
	Var[X]	0.7499	0.8750	0.9979	1.0758	0.9937

Table 3.13 Approximate values of $1 - P_0$, the probability of a non-empty repair stage for the G/G/R MRP with warm standbys ($C_M = 0.25$, $C_S = 0.25$, $C_R = 0.25$, $\theta_\alpha = 0.05$).

(A: M = 10, S = 5, N = 15)

θ_α	R : 1	2	3	5	6	8
0.2	1.00000	0.99903	0.99576	0.99555	0.99555	0.99555
0.4	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
0.6	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
0.8	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.0	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.2	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.4	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.6	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.8	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999

(B: M = 5, S = 2, N = 7)

θ_λ	R : 1	2	3	4	5	6
0.2	0.95388	0.87646	0.87538	0.87538	0.87538	0.87538
0.4	0.99998	0.99486	0.99268	0.99263	0.99263	0.99263
0.6	1.00000	0.99989	0.99955	0.99951	0.99951	0.99951
0.8	1.00000	1.00000	0.99997	0.99996	0.99995	0.99995
1.0	1.00000	1.00000	1.00000	0.99999	0.99999	0.99999
1.2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.4	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.6	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1.8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

(C: M = 3, S = 1, N = 4)

θ_λ	R : 1	2	3	4
0.2	0.63114	0.60404	0.60400	0.60400
0.4	0.95609	0.91034	0.90989	0.90989
0.6	0.99658	0.97972	0.97902	0.97902
0.8	0.99966	0.99493	0.99438	0.99437
1.0	0.99995	0.99859	0.99826	0.99825
1.2	0.99999	0.99957	0.99939	0.99938
1.4	1.00000	0.99985	0.99976	0.99976
1.6	1.00000	0.99995	0.99990	0.99990
1.8	1.00000	0.99998	0.99995	0.99995

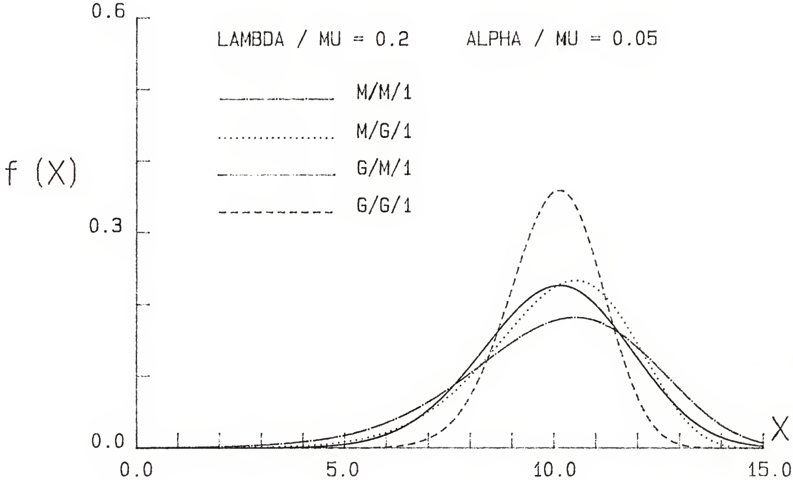


Figure 3.1 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

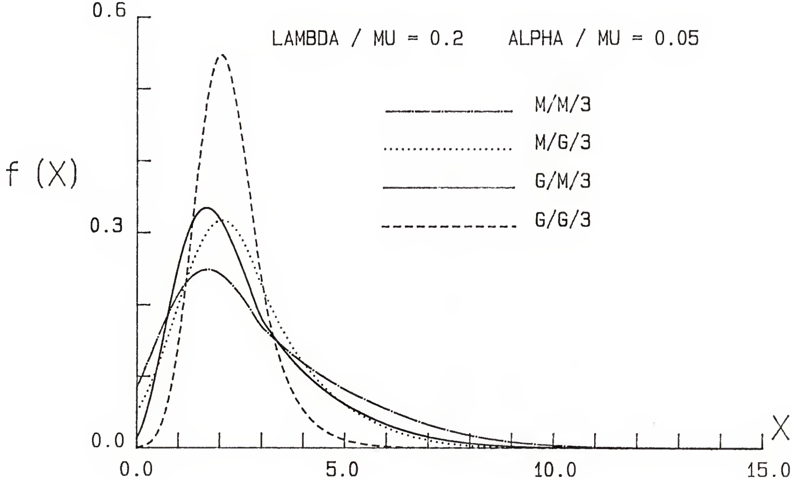


Figure 3.2 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

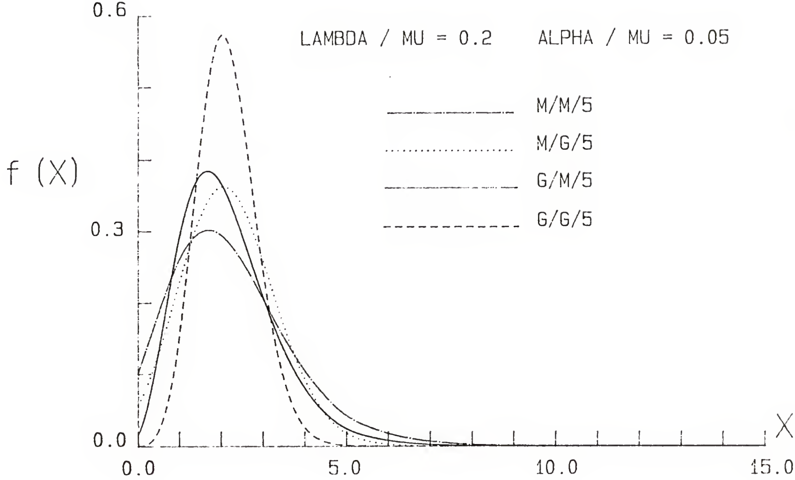


Figure 3.3 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

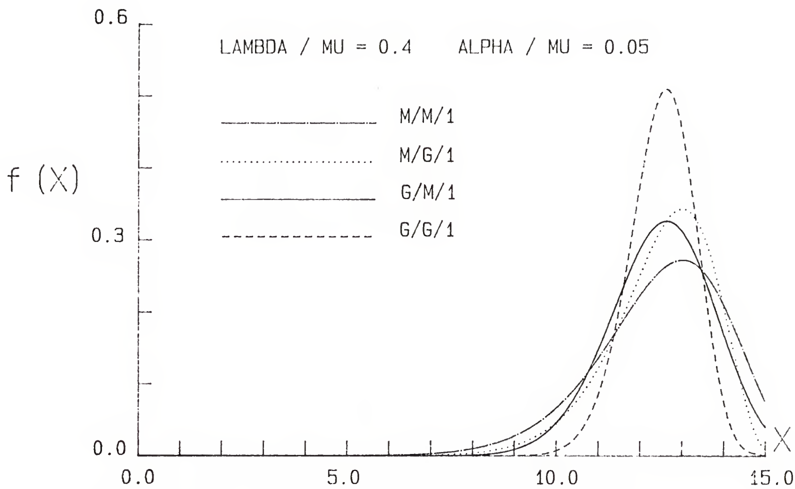


Figure 3.4 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

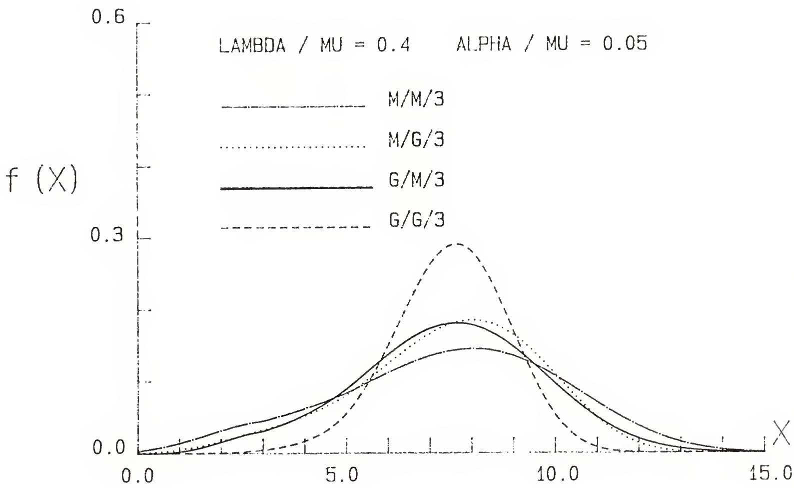


Figure 3.5 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

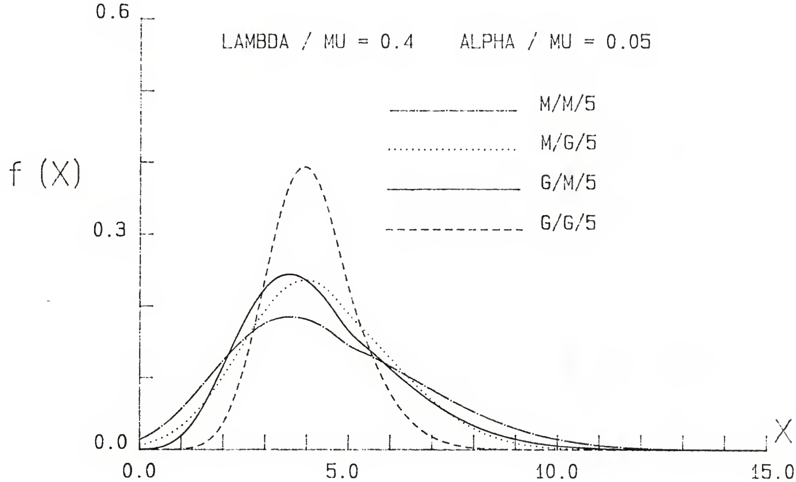


Figure 3.6 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

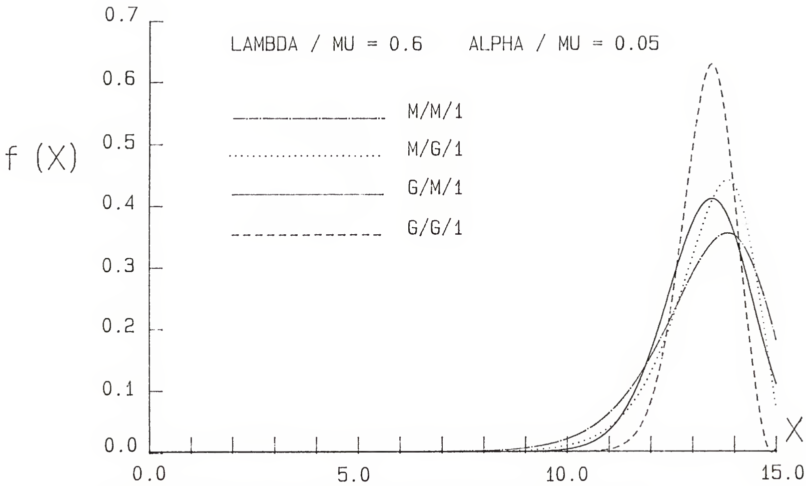


Figure 3.7 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

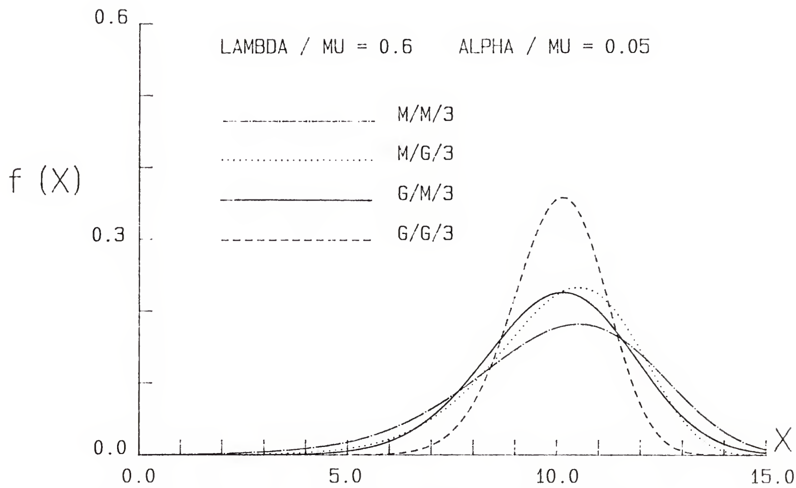


Figure 3.8 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

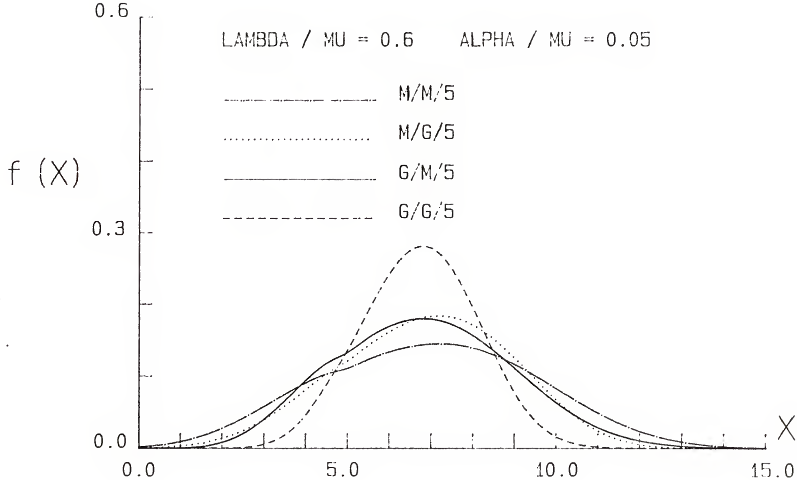


Figure 3.9 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

CHAPTER 4

SYSTEM CHARACTERISTICS AND ECONOMIC ANALYSIS OF THE MACHINE REPAIR PROBLEM

In Chapters 2 and 3, we develop the exact steady-state probability mass function (p.m.f.) P_n, and the approximate steady-state probability density function (p.d.f.) f(x) for the M/M/R and the G/G/R machine repair problems with warm standbys, respectively. The derived results P_n and f(x) will be used in this chapter to analyze system characteristics and to develop an economic model. We first introduce several measures of effectiveness such as machine availability, operative utilization, coefficients of loss for machines and repairmen, effective mean arrival rate, and expected waiting time to investigate the M/M/R and the G/G/R machine repair problems. Next, using a profit model, we determine for a numerical example the optimum number of repairmen and spares to maximize the steady-state expected profit per unit time for the M/M/R and the G/G/R machine repair problems with warm standbys.

4.1 System Characteristics of the Machine Repair Problem

In many industrial applications, it is very often desired to know specific system characteristics of the machine repair problem (MRP) such as the fraction of the total time the machines are working, or being repaired, or waiting for repair, the fraction of busy repairmen, and so on. Such characteristics are of importance particularly in evaluating the performance of the system.

System characteristics was recognized by the authors of early works on MRP. Feller [15] in his classical work, introduces the idea of coefficient of loss for machines and repairmen. Benson and Cox [6] introduce the concept of machine availability, and operative utilization for the M/M/1 model and extend it to the M/M/R model. Subsequently, several other authors use those concepts in their work such as Ashcroft [4] for the M/G/1 model, Maritas and Xirokostas [44] for the $M/E_K/R$ model, and Bunday and Scraton [9] for the G/M/R model.

Such system characteristics have not been studied for more complex situations such as the G/G/R MRP with warm standbys. We will use the approximate formulas for the steady-state p.d.f. f(x) of the number of failed machines for the G/G/R MRP to calculate the following quantities:

(1) machine availability and operative utilization;

(2) coefficients of loss for machines and repairmen;

(3) effective mean arrival rate and expected waiting time.

4.1.1 Machine Availability and Operative Utilization

The machine availability represents the fraction of the total time that the machines (operating and spare machines) running. The machine availability for the M/M/R MRP (MA) is defined by (Benson and Cox [6])

$$\text{(4.1)} \qquad MA = \sum_{n=0}^{N} (N - n)P_n \,/\, N = 1 - \{ E[N] \,/\, N \},$$

where $N = M + S$, and $E[N] = \sum_{n=0}^{N} nP_n$ is the expected number of failed machines in the system, and the steady-state solutions P_n, of n failed machines in the system, are given in equations 2.6 when $R \le S$ and equations 2.7 when $R > S$.

The operative utilization represents the fraction of busy repairmen and thus measures how effectively a group of repairmen assigned to the task is utilized. The expected number of busy repairmen is given by

$$(4.2) \qquad E[B] = R - \sum_{n=0}^{R} (R - n)P_n.$$

Then the operative utilization for the M/M/R MRP (OU) is defined by (Benson and Cox [6])

$$(4.3) \qquad OU = E[B] / R.$$

By analogy to equation 4.1, the approximate machine availability for the G/G/R MRP (MA_G) is evaluated as

$$(4.4) \qquad MA_G = 1 - \{ E[X] / N \}$$

$$= 1 - \{ \int_0^N xf(x)dx / N \},$$

where $E[X] = \int_0^N xf(x)dx$ is the expected number of failed machines in the system, and the steady-state p.d.f. f(x) is given in equations 3.24 when $R \le S$ and equations 3.25 when $R > S$.

Recall that

(i) when $R \le S$

$$f(x) = \begin{cases} f_7(x), & \text{for } 0 \le x < R, \\ f_8(x), & \text{for } R \le x \le S, \\ f_9(x), & \text{for } S < x \le N, \end{cases}$$

and

(ii) when $R > S$

$$f(x) = \begin{cases} f_{10}(x), & \text{for } 0 \le x < S, \\ f_{11}(x), & \text{for } S \le x < R, \\ f_{12}(x), & \text{for } R \le x \le N. \end{cases}$$

By analogy to equations 4.2, the expected number of busy repairmen for the G/G/R MRP is evaluated as

$$(4.5a)\quad E[B]_G = R - \int_0^R (R - x) f_7(x) dx, \text{ for } R \le S,$$

$$(4.5b)\quad E[B]_G = R - \int_0^S (R - x) f_{10}(x) dx - \int_S^R (R - x) f_{11}(x) dx, \text{ for } R > S,$$

where $f_7(x)$, $f_{10}(x)$, and $f_{11}(x)$ are given in equations 3.27a, 3.25a, and 3.25b, respectively. By analogy to equation 4.3, the approximate operative utilization for the G/G/R MRP (OU_G) is given by

$$(4.6)\qquad OU_G = E[B]_G \,/\, R.$$

Example:

We set the number of operating machines to M = 10, vary the number of spare machines S from 1 to 15, and choose R = 6 (the cases $R \le S$ and $R > S$ are considered). Exact Results of the machine availability and the operative utilization for the M/M/R MRP with warm standbys are shown in Table 4.1 and Table 4.2, respectively. Likewise, approximate results of the machine availability and the operative utilization for the G/G/R MRP warm standbys are shown in Table 4.3 and Table 4.4 and the results are depicted in Figure 4.1 and Figure 4.2, respectively. Various values of θ_λ are considered. One sees from Table 4.1 and Table 4.3 that (i) for the lower value θ_λ, the machine availability increases with the number of spare machines, (ii) for the higher value θ_λ, the machine availability decreases as the number of spare machines increases. This could conceivably happen if the number of operating machines fail fast enough without being adequately compensated than the extra number of spare machines provided. Table 4.2 and Table 4.4 indicates that for a given θ_λ, the operative utilization increases with the number of spare machines. From Table 4.1 through 4.4, as would be expected for a given

number spare machines S, the machine availability decreases with θ_λ while the operative utilization increases.

Table 4.1 The exact machine availability for the M/M/6 MRP with warm standbys (M = 10, θ_α = 0.05).

	θ_λ							
S	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
1	0.911	0.835	0.770	0.713	0.663	0.617	0.538	0.471
2	0.913	0.839	0.773	0.714	0.661	0.613	0.527	0.455
3	0.916	0.844	0.778	0.716	0.660	0.607	0.513	0.436
4	0.918	0.851	0.785	0.721	0.659	0.601	0.498	0.416
5	0.921	0.857	0.792	0.726	0.660	0.595	0.481	0.395
6	0.923	0.863	0.800	0.733	0.661	0.589	0.464	0.375
7	0.924	0.868	0.807	0.739	0.662	0.582	0.447	0.356
8	0.926	0.872	0.814	0.746	0.664	0.576	0.430	0.338
9	0.927	0.876	0.821	0.752	0.666	0.569	0.413	0.321
10	0.929	0.880	0.827	0.759	0.667	0.563	0.397	0.306
11	0.930	0.883	0.832	0.764	0.669	0.556	0.381	0.292
12	0.931	0.886	0.837	0.770	0.671	0.549	0.366	0.278
13	0.932	0.889	0.841	0.775	0.672	0.542	0.352	0.267
14	0.932	0.892	0.845	0.780	0.673	0.534	0.339	0.256
15	0.933	0.894	0.849	0.784	0.674	0.526	0.326	0.245

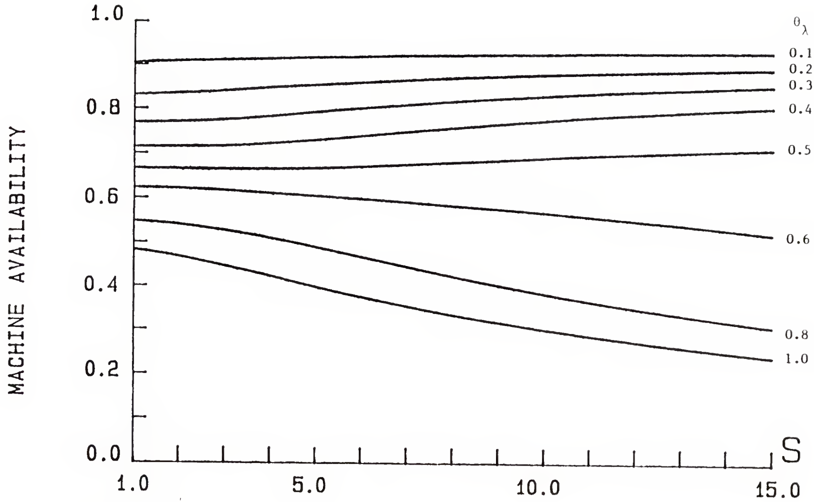


Figure 4.1 The approximate machine availability for the G/G/6 machine repair problem.

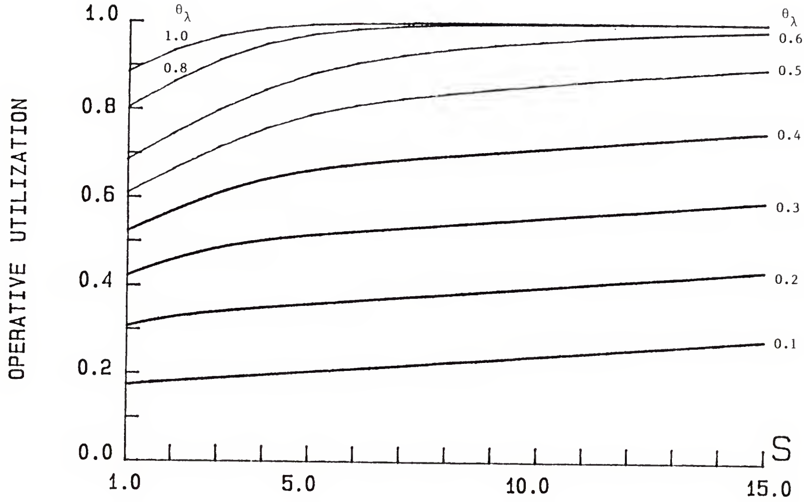


Figure 4.2 The approximate operative utilization for the G/G/6 machine repair problem.

Table 4.2 The exact operative utilization for the M/M/6 MRP with warm standbys ($M = 10$, $\theta_\alpha = 0.05$).

	θ_λ							
S	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
1	0.164	0.303	0.421	0.521	0.607	0.679	0.789	0.863
2	0.174	0.322	0.452	0.563	0.656	0.732	0.842	0.909
3	0.182	0.336	0.477	0.599	0.699	0.779	0.886	0.943
4	0.190	0.347	0.496	0.627	0.735	0.818	0.919	0.966
5	0.198	0.356	0.509	0.648	0.762	0.848	0.943	0.980
6	0.206	0.365	0.520	0.664	0.783	0.870	0.960	0.988
7	0.214	0.373	0.529	0.676	0.800	0.888	0.971	0.993
8	0.222	0.381	0.538	0.687	0.814	0.903	0.980	0.996
9	0.230	0.389	0.546	0.697	0.826	0.915	0.985	0.998
10	0.238	0.397	0.554	0.706	0.837	0.926	0.990	0.999
11	0.246	0.405	0.562	0.714	0.846	0.935	0.993	0.999
12	0.254	0.413	0.570	0.722	0.855	0.943	0.995	1.000
13	0.262	0.420	0.578	0.730	0.863	0.950	0.996	1.000
14	0.270	0.428	0.586	0.738	0.871	0.956	0.997	1.000
15	0.278	0.436	0.593	0.746	0.879	0.961	0.998	1.000

Table 4.3 The approximate machine availability for the G/G/6 MRP with warm standbys ($M = 10$, $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, $\theta_\alpha = 0.05$).

	θ_λ							
S	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
1	0.905	0.832	0.769	0.714	0.666	0.623	0.548	0.483
2	0.909	0.836	0.771	0.715	0.665	0.621	0.540	0.467
3	0.912	0.842	0.776	0.717	0.665	0.617	0.527	0.446
4	0.915	0.850	0.784	0.723	0.666	0.611	0.510	0.423
5	0.918	0.856	0.794	0.731	0.668	0.605	0.489	0.398
6	0.920	0.862	0.803	0.740	0.672	0.599	0.467	0.375
7	0.923	0.868	0.811	0.750	0.677	0.592	0.445	0.354
8	0.924	0.873	0.819	0.759	0.682	0.584	0.423	0.334
9	0.926	0.877	0.826	0.768	0.687	0.577	0.403	0.317
10	0.927	0.881	0.832	0.776	0.693	0.569	0.384	0.301
11	0.929	0.884	0.838	0.784	0.698	0.560	0.366	0.287
12	0.930	0.887	0.843	0.790	0.702	0.551	0.350	0.274
13	0.931	0.890	0.847	0.796	0.706	0.542	0.335	0.262
14	0.932	0.893	0.852	0.802	0.710	0.532	0.321	0.251
15	0.933	0.895	0.856	0.807	0.713	0.521	0.308	0.241

Table 4.4 The approximate operative utilization for the G/G/6 MRP with warm standbys ($M = 10$, $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, $\theta_\alpha = 0.05$).

S	θ_λ 0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
1	0.174	0.307	0.423	0.524	0.610	0.685	0.804	0.885
2	0.183	0.328	0.458	0.569	0.664	0.744	0.863	0.934
3	0.190	0.341	0.485	0.610	0.665	0.799	0.912	0.967
4	0.197	0.351	0.503	0.641	0.666	0.845	0.949	0.986
5	0.205	0.359	0.515	0.662	0.668	0.881	0.972	0.995
6	0.212	0.367	0.524	0.676	0.672	0.906	0.985	0.998
7	0.219	0.375	0.532	0.687	0.677	0.924	0.992	0.999
8	0.227	0.382	0.540	0.696	0.682	0.938	0.996	1.000
9	0.234	0.390	0.548	0.704	0.687	0.949	0.998	1.000
10	0.242	0.398	0.556	0.712	0.693	0.957	0.999	1.000
11	0.249	0.406	0.564	0.720	0.698	0.965	0.999	1.000
12	0.257	0.414	0.571	0.728	0.702	0.971	1.000	1.000
13	0.264	0.421	0.579	0.736	0.706	0.976	1.000	1.000
14	0.272	0.429	0.587	0.744	0.710	0.980	1.000	1.000
15	0.280	0.437	0.595	0.751	0.713	0.984	1.000	1.000

4.1.2 Coefficient of Loss for Machines and Coefficient of Loss for Repairmen

Feller [15] defines two types of coefficients of loss related to the MRP, namely,

(i) the coefficient of loss for machines (CLM)

$$\text{CLM} = \sum_{n=R}^{N} (n - R)P_n \,/\, N, \tag{4.7}$$

which represents the fraction of the total machines which are waiting in line,

(ii) the coefficient of loss for repairmen (CLR)

$$\text{CLR} = \sum_{n=0}^{R} (R - n)P_n \,/\, R, \tag{4.8}$$

which represents the fraction of idle repairmen. This is one minus the operative utilization.

Let CLM_G and CLR_G denote respectively the approximate coefficient of loss for machines and the approximate coefficient of loss for repairmen for the G/G/R MRP. Then again by analogy to equations (4.7) and (4.8), we obtain

$$(4.9a) \qquad CLM_G = \int_R^S (x - R)f_8(x)dx + \int_S^N (x - R)f_9(x)dx, \qquad \text{for } R \le S,$$

$$(4.9b) \qquad CLM_G = \int_R^N (x - R)f_{12}(x)dx \,/\, N, \qquad \text{for } R > S,$$

and

$$(4.10) \qquad CLR_G = \int_0^R (R - x)f_7(x)dx \,/\, R$$

$$= 1 - OU_G,$$

where $f_7(x)$, $f_8(x)$, $f_9(x)$, and $f_{12}(x)$ are given in equations 3.24a through 3.24c, and equation 3.25c, respectively.

4.1.3 Effective Mean Arrival Rate and Expected Waiting Time

The waiting time characteristics of MRP may be derived from a knowledge of the probability distribution function of the time spent by failed machines in the system before being repaired. Since we will only be interested in determining the expected waiting time. Little's formula may be appropriately used. However, it is necessary to first define the concept of effective mean arrival rate λ_{eff} which is defined as

$$(4.11) \qquad \lambda_{eff} = \sum_{n=0}^{N} \lambda_n P_n,$$

where λ_n represents the arrival rate to the repair facilities when the number of failed machines in the system is n; hence, it is a function of the random variable n. Thus λ_{eff} is the mathematical expectation of the λ_n's. Once we determine the effective mean arrival rate, the expected waiting time can be derived from E[*N*] by Little's formula. To do this, we note that if the system has $n \le S$ failed machines, then the population or source consists of M operating machines each with failure rate λ, and of (S - n) spare machines each with failure rate α. Therefore the failure (arrival) rate is $\lambda_n = M\lambda + (S - n)\alpha$. Likewise, if the system has $n > S$ failed machines, then the failure (arrival) rate is $\lambda_n = (N - n)\lambda$. From equation 4.11, we obtain the effective mean arrival rate for the M/M/R MRP with warm standbys given by

$$(4.12) \qquad \lambda_{eff} = \sum_{n=0}^{S} [M\lambda + (S - n)\alpha] \, P_n + \sum_{n=S+1}^{N} (N - n)\lambda \, P_n$$

$$= M\lambda + \alpha \sum_{n=0}^{S} (S - n) \, P_n - \lambda \sum_{n=S}^{N} (n - S) \, P_n .$$

It should be noted that i) when S = 0, and $\alpha = 0$, equation (4.12) reduces to the existing results for the no spares case (see equation (2.72) in Gross and Harris [22]), and ii) when $\alpha = 0$, equation (4.12) reduces to the existing results for the cold standbys case (see equation (2.78) in Gross and Harris [22]).

Let

E[W] ≡ expected waiting time in the system in the steady-state (including the service time),

$E[W_q]$ = expected waiting time in the queue in the steady-state (excluding the service time),

$E[N_q]$ = expected number of failed machines in the queue.

By Little's formula we have

$$(4.13) \qquad E[W] = E[N] \,/\, \lambda_{eff},$$

and

$$(4.14) \qquad E[W_q] = E[N_q] \,/\, \lambda_{eff}.$$

Substituting equations 4.13 and 4.14 in the general relation

$$(4.15) \qquad E[W] = E[W_q] + 1/\mu,$$

the following expressions for λ_{eff} are given by

$$(4.16) \qquad \begin{aligned} \lambda_{eff} &= \mu \, (E[N] - E[N_q]) \\ &= \mu \, E[B]. \end{aligned}$$

Thus μ E[B] which represents the actual number of repair completions in the repair facilities, is equivalent to the effective mean arrival rate.

From equations 4.3 and 4.16, we can express the operative utilization in terms of λ_{eff}, R and μ,

$$(4.17) \qquad OU = E[B] \,/\, R = \lambda_{eff} \,/\, R\mu.$$

In this form the OU is sometime called the server utilization.

By analogy to equation 4.16, we obtain the effective mean arrival rate for the G/G/R MRP (λ_{eff}) as follows:

(i) for $R \leq S$

$$(4.18) \qquad \lambda_{effG} = \mu \, [\, R - \int_0^R (R - x) f_7(x)dx \,],$$

and (ii) for R > S

$$(4.19) \qquad \lambda_{effG} = \mu \left[R - \int_0^S (R - x) f_{10}(x) dx - \int_S^R (R - x) f_{11}(x) dx \right].$$

By analogy to equation 4.13, we obtain the expected waiting time for the G/G/R MRP ($E[W]_G$)

$$(4.20) \qquad E[W]_G = E[X] / \lambda_{effG}.$$

Example:

Here we select M = 10, S = 5, $\theta_\alpha = 0.05$, vary the number of repairmen R ($3 \le R \le 8$) and θ_λ ($0.2 \le \theta_\lambda \le 2.0$). Exact results of the effective mean arrival rate (λ_{eff}) and the expected waiting time (E[W]) for the M/M/R MRP with warm standbys are shown in Table 4.5. One sees from Table 4.5 that

(i) for R = 3, $\theta_\lambda = 0.6$, we find that $E[N] = 9.98$. Thus on the average about 10 out of the 15 machines will be either in repair or waiting for repair. The average time spent in the repair facility is about 3.33;

(ii) by doubling the number of repairmen to R = 6, we find $E[N] = 6.08$. Thus on the average about 6 out of the 15 machines will be either in repair or waiting for repair. The average time spent in the repair facility is cut by more than half to 1.20.

Of course, the expected number of failed machines and the time lost during repairs can be used to determine the number of repairmen.

As would be expected (i) for a given θ_λ, the expected waiting time decreases as the number of repairmen increases, and (ii) for a given

number of repairmen, R, the expected waiting time increases with the value of θ_λ.

Table 4.5 Exact of the effective mean arrival rate and the expected waiting time for the M/M/R MRP with warm standbys ($M = 10$, $S = 5$, $\theta_\alpha = 0.05$).

	R							
	3		5		6		8	
θ_λ	λ_{eff}	E[W]	λ_{eff}	E[W]	λ_{eff}	E[W]	λ_{eff}	E[W]
0.2	2.077	1.346	2.136	1.018	2.139	1.004	2.139	1.000
0.3	2.689	1.916	3.031	1.074	3.056	1.019	3.066	1.001
0.4	2.925	2.529	3.787	1.180	3.887	1.057	3.933	1.004
0.5	2.983	2.999	4.328	1.326	4.573	1.117	4.711	1.011
0.6	2.996	3.331	4.656	1.487	5.086	1.195	5.381	1.022
0.7	2.999	3.570	4.831	1.644	5.436	1.284	5.940	1.040
0.8	3.000	3.750	4.918	1.785	5.661	1.375	6.397	1.061
0.9	3.000	3.889	4.960	1.906	5.798	1.462	6.763	1.087
1.0	3.000	4.000	4.980	2.008	5.880	1.544	7.051	1.115
1.2	3.000	4.167	4.995	2.169	5.956	1.683	7.447	1.176
1.6	3.000	4.375	4.999	2.375	5.993	1.878	7.811	1.295
2.0	3.000	4.500	5.000	2.500	5.999	2.001	7.932	1.391

Approximate results of the effective mean arrival rate (λ_{effG}) and the expected waiting time ($E[W]_G$) for the G/G/R MRP with warm standbys are shown in Table 4.6, for two sets of values, namely

(i) $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$;

(ii) $C_M = 0.25$, $C_S = 0.1$, $C_R = 0.25$.

One sees from Table 4.6 that the square coefficients of variation C_M, C_S, and C_R are not affected significantly for the effective mean arrival rate and the expected waiting time.

Table 4.6 Approximate of the effective arrival rate and the expected waiting time for the G/G/R MRP with warm standbys (M = 10, S = 5, θ_α = 0.05).

		R							
		3		5		6		8	
θ_λ		λ_{effG}	$E[W]_G$	λ_{effG}	$E[W]_G$	λ_{effG}	$E[W]_G$	λ_{effG}	$E[W]_G$
0.2	(i)	2.131	1.149	2.154	1.002	2.154	1.000	2.154	1.000
	(ii)	2.138	1.047	2.143	1.000	2.143	1.000	2.143	1.000
0.3	(i)	2.807	1.731	3.080	1.020	3.088	1.003	3.089	1.000
	(ii)	2.887	1.612	3.093	1.003	3.094	1.000	3.094	1.000
0.4	(i)	2.984	2.486	3.916	1.088	3.974	1.016	3.988	1.000
	(ii)	2.999	2.492	3.993	1.036	4.019	1.003	4.021	1.000
0.5	(i)	2.999	2.997	4.515	1.226	4.730	1.053	4.800	1.001
	(ii)	3.000	3.000	4.653	1.157	4.833	1.019	4.859	1.000
0.6	(i)	3.000	3.333	4.825	1.409	5.288	1.120	5.497	1.005
	(ii)	3.000	3.333	4.929	1.363	5.433	1.069	5.564	1.000
0.7	(i)	3.000	3.571	4.945	1.594	5.640	1.211	6.080	1.012
	(ii)	3.000	3.571	4.990	1.575	5.781	1.160	6.149	1.002
0.8	(i)	3.000	3.749	4.984	1.757	5.832	1.314	6.558	1.025
	(ii)	3.000	3.750	4.999	1.750	5.932	1.277	6.637	1.007
0.9	(i)	3.000	3.888	4.995	1.891	5.926	1.417	6.942	1.044
	(ii)	3.000	3.889	5.000	1.889	5.982	1.396	7.041	1.018
1.0	(i)	3.000	3.998	4.998	2.001	5.968	1.512	7.241	1.069
	(ii)	3.000	4.000	5.000	2.000	5.996	1.502	7.362	1.037
1.2	(i)	3.000	4.164	5.000	2.167	5.994	1.669	7.634	1.131
	(ii)	3.000	4.167	5.000	2.167	6.000	1.667	7.764	1.099
1.6	(i)	3.000	4.368	5.000	2.374	6.000	1.875	7.927	1.267
	(ii)	3.000	4.375	5.000	2.375	6.000	1.875	7.982	1.254
2.0	(i)	3.000	4.489	5.000	2.498	6.000	1.999	7.986	1.378
	(ii)	3.000	4.499	5.000	2.500	6.000	2.000	7.999	1.375

4.2 Economic Analysis of the Machine Repair Problem

The M/M/R MRP with cold standbys were first considered by Taylor and Jackson [63], while the incorporation of a cost model for cold standbys was studied by Hilliard [30], Gross et al. [23, 24], and for a warm-standby model by Albright [1, 2], and Sivazlian and Wang [57].

The decision variables which are considered in an economic model have for the most part been either the number of repairmen, or the number of spares, or the rate of failure, or the rate of repair. The only authors that have so far accounted for both the number of spares and the number of repairmen as decision variables have been Hilliard [30], Gross et al. [23, 24].

We develop an expected profit function per unit time in which the number of repairmen and the number of spares are decision variables. Our objective is (i) to determine the optimal values of the number of repairmen and spares so as to maximize an expected profit function, and (ii) to determine several systems characteristics under optimal operating conditions for the M/M/R and G/G/R machine repair problems with warm standbys.

4.2.1 Economic Analysis of the M/M/R Machine Repair Problem

We develop an expected profit function per unit time in which R and S are decision variables and determine the optimum value of (R, S), say (R^*, S^*), so as to maximize this function.

We first provide the following four expressions E[O], E[S], E[B], and E[I] for the M/M/R MRP which are required for this economic analysis.

In state n, let

$M_n \equiv$ the number of operating machines,

$S_n \equiv$ the number of spare machines,

$R_n \equiv$ the number of repairmen.

The values of M_n, S_n, and R_n for $R \le S$ and $R > S$ are shown in Table 4.7.

Table 4.7 The Values of M_n, S_n, and R_n ($N = M + S$).

(i) $R \le S$				(ii) $R > S$			
Range of n	M_n	S_n	R_n	Range of n	M_n	S_n	R_n
$0 \le n < R$	M	S - n	n	$0 \le n \le S$	M	S - n	n
$R \le n \le S$	M	S - n	R	$S < n \le R$	N - n	0	n
$S < n \le N$	N - n	0	R	$R < n \le N$	N - n	0	R

Let

$E[O] \equiv$ the expected number of operating machines in the system,

$E[S] \equiv$ the expected number of spare machines in the system acting as standbys.

Using Table 4.7 we obtain

$$(4.21) \qquad E[O] = M - \sum_{n=S}^{N} (n - S)P_n,$$

$$(4.22) \qquad E[S] = \sum_{n=0}^{S} (S - n)P_n,$$

where P_n is given in Chapter 2 for the M/M/R MRP with warm standbys.

Also, the expected number of idle repairmen, $E[I]$, is given by

(4.23) $\quad E[I] = R - E[B],$

where E[B] is given in equation 4.2.

Let

$p \equiv$ revenue per unit time when one machine is in an operating state,

$C_1 \equiv$ cost per unit time when one machine is in an operating state,

$C_2 \equiv$ cost per unit time when one machine is functioning as a warm standby,

$C_3 \equiv$ cost per unit time when one repairman is servicing,

$C_4 \equiv$ cost per unit time when one repairman is idle.

Then the expected profit per unit time, F(R, S), for the M/M/R MRP is given by

(4.24) $$F(R, S) = (p - C_1)\ E[O] - C_2\ E[S] - C_3\ E[B] - C_4\ E[I].$$

The optimum values of (R, S), say (R^*, S^*), can be determined so as to maximize the function F(R, S), namely,

(4.25) $$\max_{R,\ S} \{\ F(R, S) = (p - C_1)\ E[O] - C_2\ E[S] - C_3\ E[B] - C_4\ E[I]\ \},$$

where E[B], E[O], E[S], and E[I] are given in equations 4.2, 4.21, 4.22 and 4.23, respectively.

It is either difficult or completely impossible to obtain usable analytical results for the optimum value (R^*, S^*). This is due to the fact that R ans S are discrete quantities and the profit function is highly nonlinear and complicated. In this situations one may use computational approximations to obtain potentially useful results. Computationally efficient procedure may be developed which is based on

the necessary conditions for a maximum for the profit function. The following conditions prevail:

(i) the optimal number of spares for S, S^*

$$F(R, S) \geq F(R, S-1) \quad \text{and} \quad F(R, S) \geq F(R, S+1),$$

or $\max_S \{ F(R, S) \} = F(R, S^*)$,

(ii) the optimal number of repairmen for R, R^*

$$F(R, S^*) \geq F(R-1, S^*) \quad \text{and} \quad F(R, S^*) \geq F(R+1, S^*),$$

or $\max_R \{ F(R, S^*) \} = F(R^*, S^*)$.

Also, one could perhaps use the implicit enumeration optimization algorithm, i.e., the Lawler-Bell algorithm, to solve the problem. Gross et al. [24] have applied this approach as an efficient procedure to solve an optimization problem in a multi-echelon repairable item provisioning system. Alternatively, the optimum value (R^*, S^*) may be found by direct substitution of successive values of R and S into the profit function until the maximum value of F(R, S) is identified.

Example:

$M = 10$, $1 \leq S \leq 15$, $1 \leq R \leq 15$, $\theta_\lambda = 0.2$, $\theta_\alpha = 0.05$, $p = \$350/\text{day}$,

$C_1 = \$75/\text{day}$, $C_2 = \$50/\text{day}$, and $C_3 = C_4 = \$100/\text{day}$.

The expected profit F(R, S) for the M/M/R MRP with warm standbys is shown in Table 4.8 for different values of R and S. We note that a maximum expected profit per day of \$2277.11 is achieved at $R^* = 3$ and $S^* = 4$.

Table 4.8 The expected profit F(R, S) for the M/M/R MRP with warm standbys (M = 10, θ_α = 0.05, θ_λ = 0.2).

S \ R	1	2	3	4	5
1	1261.72	2011.08	2124.59	2069.07	1977.68
2	1267.15	2083.60	2225.78	2179.10	2090.63
3	1269.57	2127.47	2267.60	2219.90	2132.52
4	1270.61	2153.70	2277.11	2216.04	2126.66
5	1271.06	2168.67	2267.96	2191.11	2096.36
6	1271.25	2176.33	2247.62	2155.86	2056.22
7	1271.32	2179.24	2220.40	2115.42	2012.02
8	1271.35	2179.15	2188.92	2072.33	1966.10
9	1271.36	2177.25	2154.82	2027.89	1919.47
10	1271.37	2174.38	2119.15	1982.77	1872.52
11	1271.37	2171.10	2082.64	1937.34	1825.47
12	1271.37	2167.81	2045.76	1891.80	1778.38
13	1271.37	2164.72	2008.89	1846.26	1731.30
14	1271.37	2161.99	1972.28	1800.79	1684.26
15	1271.37	2159.67	1936.14	1755.44	1637.25

S \ R	6	7	8	9	10
1	1879.07	1779.25	1679.27	1579.27	1479.27
2	1992.80	1893.12	1793.16	1693.16	1593.16
3	2035.27	1935.77	1835.84	1735.84	1635.84
4	2029.45	1930.05	1830.15	1730.17	1630.17
5	1998.39	1898.97	1799.09	1699.11	1599.11
6	1956.68	1857.04	1757.15	1657.17	1557.17
7	1911.25	1811.20	1711.26	1611.28	1511.28
8	1864.50	1764.16	1664.12	1564.13	1464.13
9	1817.27	1716.75	1616.65	1516.63	1416.63
10	1769.87	1669.22	1569.07	1469.04	1369.04
11	1722.41	1621.65	1521.47	1421.44	1321.43
12	1674.94	1574.07	1473.86	1373.82	1273.81
13	1627.47	1526.49	1426.26	1326.20	1226.19
14	1580.02	1478.92	1378.65	1278.59	1178.57
15	1532.57	1431.35	1331.04	1230.97	1130.96

The maximum expected profit F(R, S), the values of E[O], E[S], E[N], E[B], E[I], effective mean arrival rate (λ_{eff}), expected waiting time (E[W]), machine availability (MA), operative utilization (OU), coefficient of loss for machines (CLM), coefficient of loss for

repairmen (CLR), and P_o at the optimum values (R^*, S^*) are shown in Table 4.9 for for $\theta_\alpha = 0.05$, and for $0.2 \le \theta_\lambda \le 1.0$.

Table 4.9 Steady-state system characteristics of the M/M/R MRP with warm standbys under optimal operating conditions.

θ_λ	0.2	0.4	0.6	0.8	1.0
(R^*, S^*)	(3, 4)	(5, 6)	(6, 10)	(8, 12)	(10, 13)
$F(R^*, S^*)$	2277.11	2015.50	1786.70	1586.29	1385.69
E[O]	9.681	9.470	9.075	9.117	9.084
E[S]	1.702	1.776	2.178	2.418	2.251
E[N]	2.617	4.754	8.747	10.465	11.665
E[B]	2.021	3.877	5.554	7.415	9.197
E[I]	0.979	1.123	0.446	0.585	0.803
λ_{eff}	2.021	3.877	5.554	7.415	9.197
E[W]	1.295	1.226	1.575	1.411	1.268
MA	0.813	0.703	0.563	0.524	0.493
OU	0.674	0.775	0.926	0.927	0.920
CLM	0.043	0.055	0.160	0.139	0.107
CLR	0.326	0.225	0.074	0.073	0.080
$1 - P_N$	0.999999	0.999996	0.999966	0.999968	0.999967

Table 4.9 provides values for $1 - P_N$ which is the probability that at least one machine is operating. This may be taken as a measure of system availability if one considers the system to be in a failed state when there are no machines in operation.

From Table 4.9, as would be expected, one observes that:

(i) $F(R^*, S^*)$ decreases as θ_λ increases;

(ii) the optimum value of (R, S), (R^* ,S^*) increases in θ_λ. Note for example that it may sometimes be desirable to have more spares than operating machines.

Also exhibited are the optimum values of R^* for fixed S in Table 4.10, and conversely the optimum values of S^* for fixed R in Table 4.11.

Table 4.10 The optimum number of repairmen for various values of θ_λ, when the number of spares is fixed.

		The number of spares (S)														
θ_λ		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.2	R^*	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2
0.4	R^*	4	5	5	5	5	5	5	5	5	4	4	4	4	4	4
0.6	R^*	5	5	6	6	6	7	7	7	6	6	6	6	6	6	6
0.8	R^*	6	6	6	7	7	8	8	8	8	8	8	8	8	8	8
1.0	R^*	6	6	7	7	8	8	9	9	9	10	10	10	10	10	9

Table 4.11 The optimum number of spares for various values of θ_λ, when the number of repairmen is fixed.

		The number of repairmen (R)														
θ_λ		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.2	S^*	15	7	4	3	3	3	3	3	3	3	3	3	3	3	3
0.4	S^*	15	15	15	9	7	6	6	6	6	6	6	6	6	6	6
0.6	S^*	15	15	15	15	14	10	9	8	8	8	8	8	8	8	8
0.8	S^*	15	15	15	15	15	15	14	12	11	10	10	10	10	10	10
1.0	S^*	15	15	15	15	15	15	15	15	15	13	12	12	12	12	12

4.2.2 Economic Analysis of the G/G/R Machine Repair Problem

Next, we develop the following four expressions $E[O]_G$, $E[S]_G$, $E[B]_G$, and $E[I]_G$ for the G/G/R MRP which are required for this economic analysis.

By analogy to equations 4.21, 4.22, 4.2, and 4.23, we obtain the respective expressions $E[O]_G$, $E[S]_G$, $E[B]_G$, and $E[I]_G$, for the G/G/R MRP as follows:

(i) when $R \leq S$

$$E[O]_G = M - \int_S^{M+S} (x - S) f_9(x)dx, \tag{4.26}$$

$$E[S]_G = \int_0^R (S - x) f_7(x)dx + \int_R^S (S - x) f_8(x)dx, \tag{4.27}$$

$$E[B]_G = R - \int_0^R (R - x) f_7(x)dx, \tag{4.28}$$

$$E[I]_G = \int_0^R (R - x) f_7(x)dx = R - E[B]_G, \tag{4.29}$$

where $f_7(x)$, $f_8(x)$, and $f_9(x)$ are given in equations 3.24a through 3.24c, respectively,

and (ii) when $R > S$

$$E[O]_G = M - \int_S^R (x - S) f_{11}(x)dx - \int_R^{M+S} (x - S) f_{12}(x)dx, \tag{4.30}$$

$$E[S]_G = \int_0^S (S - x) f_{10}(x)dx, \tag{4.31}$$

$$(4.32) \qquad E[B]_G = R - \int_0^S (R - x) f_{10}(x)dx - \int_S^R (R - x) f_{11}(x)dx,$$

$$(4.33) \qquad E[I]_G = \int_0^S (R - x) f_{10}(x)dx + \int_S^R (R - x) f_{11}(x)dx = R - E[B]_G,$$

where $f_{10}(x)$, $f_{11}(x)$, and $f_{12}(x)$ are given in equations 3.25a through 3.25c, respectively.

By analogy to equation 4.24, the expected profit per unit time, F(R, S), for the G/G/R MRP is given by

$$(4.34) \qquad F(R, S) = (p - C_1)\ E[O]_G - C_2\ E[S]_G - C_3\ E[B]_G - C_4\ E[I]_G.$$

The optimum values of (R, S), say (R^*, S^*), can be determined so as to maximize the function F(R, S), namely,

$$(4.35) \qquad \max_{R,S} \{F(R, S) = (p - C_1)\ E[O]_G - C_2\ E[S]_G - C_3\ E[B]_G - C_4\ E[I]_G\},$$

where $E[O]_G$, $E[S]_G$, $E[B]_G$, and $E[I]_G$ are given in equations 4.26 through 4.29 for $R \leq S$, and are given in equations 4.30 through 4.33 for $R > S$, respectively.

The optimum value (R^*, S^*) of (R, S) is obtained by differentiating respectively F(R, S) with respect to R and S and setting the results to zero, yielding

$$(4.36) \qquad \frac{\partial F(R, S)}{\partial R} = 0,$$

and

$$(4.37) \qquad \frac{\partial F(R, S)}{\partial S} = 0,$$

or

$$(4.38)\qquad (p - C_1)\frac{\partial E[O]_G}{\partial R} - C_2\frac{\partial E[S]_G}{\partial R} - C_3\frac{\partial E[B]_G}{\partial R} - C_4\frac{\partial E[I]_G}{\partial R} = 0,$$

and

$$(4.39)\qquad (p - C_1)\frac{\partial E[O]_G}{\partial S} - C_2\frac{\partial E[S]_G}{\partial S} - C_3\frac{\partial E[B]_G}{\partial S} - C_4\frac{\partial E[I]_G}{\partial S} = 0.$$

The differentiations of $E[O]_G$, $E[S]_G$, $E[B]_G$, and $E[I]_G$ with respect to R and S are based on the Leibnitz's rule for differentiations of integrals for $R \le S$ which are shown as follows:

$$(4.40)\qquad \frac{\partial E[O]_G}{\partial R} = -\int_S^{M+S} (x-S)\frac{\partial f_9(x)}{\partial R}\,dx,$$

$$(4.41)\qquad \frac{\partial E[S]_G}{\partial R} = \int_0^R (S-x)\frac{\partial f_7(x)}{\partial R}\,dx + \int_R^S (S-x)\frac{\partial f_8(x)}{\partial R} + (S-R)f_7(R) - (S-R)f_8(R),$$

$$(4.42)\qquad \frac{\partial E[B]_G}{\partial R} = 1 - \left[\int_0^R f_7(x)dx + \int_0^R (R-x)\frac{\partial f_7(x)}{\partial R}\,dx\right],$$

$$(4.43)\qquad \frac{\partial E[I]_G}{\partial R} = 1 - \frac{\partial E[B]_G}{\partial R},$$

$$(4.44)\qquad \frac{\partial E[O]_G}{\partial S} = -\int_S^{M+S} \left[(x-S)\frac{\partial f_9(x)}{\partial S} - f_9(x)\right]dx - M\,f_9(M+S),$$

$$(4.45)\qquad \frac{\partial E[S]_G}{\partial S} = \int_0^R \left[f_7(x) + (S-x)\frac{\partial f_7(x)}{\partial S}\right]dx + \int_R^S \left[f_8(x) + (S-x)\frac{\partial f_8(x)}{\partial S}\right]dx - (S-R)f_8(R),$$

$$(4.46)\qquad \frac{\partial E[B]_G}{\partial S} = \int_0^R (R-x)\frac{\partial f_7(x)}{\partial S}\,dx,$$

$$(4.47) \qquad \frac{\partial E[I]_G}{\partial S} = \frac{\partial E[B]_G}{\partial S}.$$

Following the same procedures, we can obtain the differentiations of $E[O]_G$, $E[S]_G$, $E[B]_G$, and $E[I]_G$ with respect to R and S for the other case when $R > S$. As may be noted, it is very difficult to solve equations 4.38 and 4.39 simultaneously by any reasonable numerical methods. Even the use of computational approximations to obtain the optimum value (R, S) becomes an arduous task. This is due to the fact that there are several integrals and partial derivatives in equations 4.38 and 4.39. Although, a number of techniques may be used to obtain the optimum value (R^*, S^*), it was decided to use direct substitution of successive values of R and S into the profit function until the maximum value of F(R, S) is obtained. This was done because the objective of this chapter is to show how approximate systems characteristics can be obtained for a general G/G/R MRP rather than develop efficient computational techniques to obtain the optimal values. We now consider the same example as in Section 4.2.1. The expected profit F(R, S) for the G/G/R MRP with warm standbys is shown in Table 4.12 for different values of R and S. We note that a maximum expected profit per day of \$2346.70 is achieved at $R^* = 3$ and $S^* = 3$.

Table 4.12 The expected profit F(R, S) for the G/G/R MRP with warm standbys ($C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, $M = 10$, $\theta_\alpha = 0.05$, $\theta_\lambda = 0.2$).

S \ R	1	2	3	4	5
1	1273.75	2119.64	2180.72	2097.66	1999.13
2	1274.60	2209.99	2308.86	2231.97	2134.37
3	1274.79	2254.52	2346.70	2267.97	2170.75
4	1274.84	2275.93	2336.77	2246.75	2148.60
5	1274.84	2284.87	2307.55	2206.58	2106.54
6	1274.84	2287.04	2270.02	2161.29	2060.00
7	1274.84	2285.71	2228.82	2114.57	2012.60
8	1274.84	2282.87	2186.05	2067.45	1965.03
9	1274.84	2279.64	2142.71	2020.25	1917.43
10	1274.84	2276.68	2099.32	1973.05	1869.83
11	1274.84	2274.27	2056.15	1925.90	1822.25
12	1274.84	2272.46	2013.40	1878.81	1774.68
13	1274.84	2271.21	1971.21	1831.78	1727.12
14	1274.84	2270.40	1929.69	1784.83	1679.58
15	1274.84	2269.90	1888.97	1737.96	1632.05
S \ R	**6**	**7**	**8**	**9**	**10**
1	1899.22	1799.22	1699.23	1599.23	1499.23
2	2034.55	1934.57	1834.57	1734.57	1634.57
3	2071.02	1971.04	1871.04	1771.04	1671.04
4	2048.87	1948.90	1848.90	1748.90	1648.90
5	2006.67	1906.69	1806.69	1706.69	1606.69
6	1959.90	1859.91	1759.91	1659.91	1559.91
7	1912.37	1812.35	1712.35	1612.35	1512.35
8	1864.73	1764.70	1664.70	1564.70	1464.70
9	1817.08	1717.04	1617.04	1517.04	1417.04
10	1769.43	1669.39	1569.38	1469.38	1369.38
11	1721.78	1621.73	1521.73	1421.73	1321.73
12	1674.14	1574.08	1474.08	1374.08	1274.08
13	1626.51	1526.44	1426.43	1326.43	1226.43
14	1578.88	1478.79	1378.78	1278.78	1178.78
15	1531.25	1431.15	1331.14	1231.14	1131.14

The expected profit F(R, S) are shown in Figure 4.3 for $\theta_\lambda = 0.4$ and Figure 4.4 for $\theta_\lambda = 1.0$, respectively. The maximum expected profit F(R, S), the values of $E[O]_G$, $E[S]_G$, $E[X]_G$, $E[B]_G$, $E[I]_G$, effective mean arrival rate (λ_{effG}), expected waiting time ($E[W]_G$), machine

availability (MA_G), operative utilization (OU_G), coefficient of loss for machines (CLM_G), coefficient of loss for repairmen (CLR_G) and P_0 at the optimum values (R^*, S^*) for $0.2 \le \theta_\lambda \le 1.0$ are shown in Table 4.13 for $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, and are shown in Table 4.14 for $C_M = 0.25$, $C_S = 0.04$, $C_R = 0.25$.

Table 4.13 Steady-state system characteristics of the G/G/R MRP with warm standbys under optimal operating conditions ($C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$).

θ_λ	0.2	0.4	0.6	0.8	1.0
(R^*, S^*)	(3, 3)	(5, 6)	(6, 9)	(8, 11)	(10, 12)
$F(R^*, S^*)$	2346.70	2102.47	1891.18	1690.63	1489.75
$E[O]_G$	9.802	9.781	9.351	9.381	9.336
$E[S]_G$	0.977	1.744	1.609	1.781	1.555
$E[X]_G$	2.221	4.475	8.040	9.838	11.109
$E[B]_G$	2.023	4.000	5.691	7.594	9.414
$E[I]_G$	0.977	1.000	0.309	0.406	0.586
λ_{effG}	2.023	4.000	5.691	7.594	9.414
$E[W]_G$	1.098	1.119	1.413	1.296	1.180
MA_G	0.829	0.720	0.577	0.532	0.495
OU_G	0.674	0.800	0.949	0.949	0.941
CLM_G	0.015	0.030	0.124	0.107	0.077
CLR_G	0.326	0.200	0.051	0.051	0.059
P_0	0.03225	0.00091	0.00002	0.00000	0.00000
$1 - P_N$	1.00000	1.00000	1.00000	1.00000	1.00000

Table 4.14 Steady-state system characteristics of the G/G/R MRP with warm standbys under optimal operating conditions (C_M = 0.25, C_S = 0.04, C_R = 0.25).

θ_λ	0.2	0.4	0.6	0.8	1.0
(R^*, S^*)	(3, 3)	(4, 6)	(6, 8)	(8, 10)	(10, 12)
$F(R^*, S^*)$	2385.76	2165.26	1966.18	1765.99	1565.27
$E[O]_G$	9.942	9.501	9.531	9.552	9.569
$E[S]_G$	0.963	0.952	1.096	1.217	1.324
$E[X]_G$	2.095	5.547	7.373	9.231	11.107
$E[B]_G$	2.037	3.848	5.773	7.703	9.635
$E[I]_G$	0.963	0.152	0.227	0.297	0.365
λ_{effG}	2.023	4.000	5.691	7.594	9.414
$E[W]_G$	1.098	1.119	1.413	1.296	1.180
MA_G	0.839	0.653	0.590	0.538	0.495
OU_G	0.679	0.962	0.962	0.963	0.964
CLM_G	0.015	0.030	0.124	0.107	0.077
CLR_G	0.326	0.200	0.051	0.051	0.059
P_0	0.00136	0.00000	0.00000	0.00000	0.00000
$1 - P_N$	1.00000	1.00000	1.00000	1.00000	1.00000

Tables 4.13 and 4.14 provide values for $1 - P_N$ which is the probability that at least one machine is operating. Note that in Table 4.13 and Table 4.14, $1 - P_0$ is very close 1.0, thus justifying the assumption of heavy traffic approximation. This may be taken as a measure of system availability if one considers the system to be in a failed state when there are no machines in operation. From Tables 4.13 and 4.14, as would be expected, we observe that:

(i) $F(R^*, S^*)$ decreases as θ_λ increases;

(ii) the optimum value of (R, S), (R^*, S^*) increases in θ_λ.

Note for example that it may sometimes be desirable to have more spares than operating machines.

We have also exhibited the optimum values of R^* for fixed S in Table 4.15, and conversely the optimum values of S^* for fixed R in Table 4.16.

Table 4.15 The optimum number of repairmen for various values of θ_λ, when the number of spares is fixed ($C_M = 0.5$, $C_S = 0.2$, $C_S = 0.5$).

		The number of spares (S)														
θ_λ		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.2	R^*	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
0.4	R^*	4	4	5	5	5	5	4	4	4	4	4	4	4	4	4
0.6	R^*	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6
0.8	R^*	5	6	6	7	7	8	8	8	8	8	8	8	8	8	8
1.0	R^*	6	6	7	7	8	8	9	9	10	10	10	10	10	10	10

Table 4.16 The optimum number of spares for various values of θ_λ, when the number of repairmen is fixed ($C_M = 0.5, C_S = 0.2, C_R = 0.5$).

		The number of repairmen (R)														
θ_λ		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.2	S^*	15	6	3	3	3	3	3	3	3	3	3	3	3	3	3
0.4	S^*	15	15	15	6	5	5	5	5	5	5	5	5	5	5	5
0.6	S^*	15	15	15	15	14	9	8	7	7	7	7	7	7	7	7
0.8	S^*	15	15	15	15	15	15	13	11	10	10	10	10	10	10	10
1.0	S^*	15	15	15	15	15	15	15	15	14	12	12	12	12	12	12

It should be noted that (i) the maximization of the expected profit function may be shown to be equivalent to the minimization of an expected cost function; and (ii) the incorporation of an extra cost term proportional to the expected number of failed machines in the system would not have changed the structure of the profit model. Since $E[N] = N - E[O] - E[S]$ for the M/M/R MRP, the only parameters that would have consequently been affected in the expected profit function would have been C_1 and C_2. This property can be utilized for the G/G/R MRP. The analytic study of the behavior of the expected profit functions would have been an arduous task to undertake since the decision variables R and S and the profit functions for the M/M/R and G/G/R machine repair problems appear in an expression which is highly nonlinear and complicated. The numerical results obtained suggest however that F(R, S) achieves a global maximum.

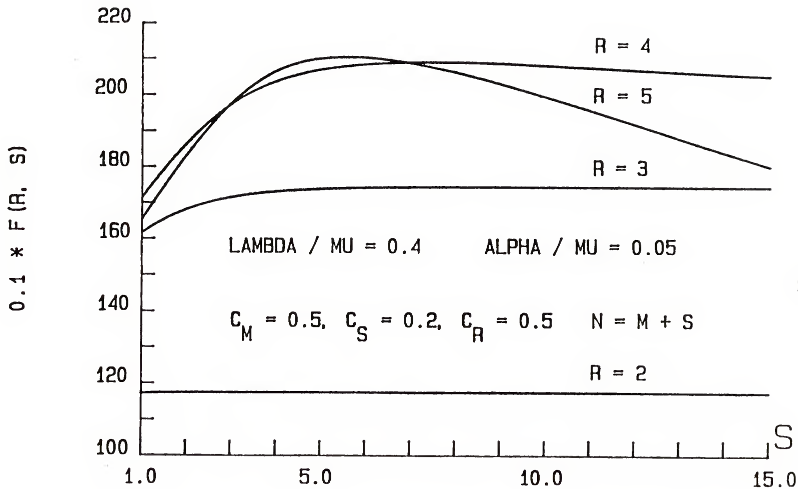


Figure 4.3 The approximate expected profit F(R,S) as a function of the number of spares, for the G/G/R machine repair problem.

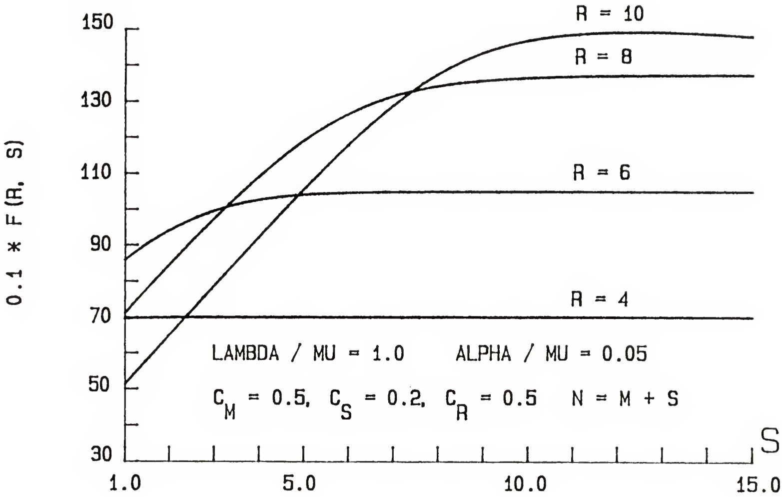


Figure 4.4 The approximate expected profit F(R,S) as a function of the number of spares for the G/G/R machine repair problem.

CHAPTER 5

COMPARATIVE ANALYSIS FOR THE MACHINE REPAIR PROBLEM WITH NO SPARES

In this chapter, we restrict ourselves to the comparative analysis of the MRP with no spares. Exact steady-state solutions of the MRP with no spares are obtained for (i) the M/M/R model by Feller [14], Barlow [5], Bhat [8], (ii) the M/G/1 and M/D/1 models by Ashcroft [4], (iii) the $M/E_k/1$ model by Benson and Cox [6], (iv) the $M/E_k/R$ model by Maritas and Xirokostas [44], and (v) the G/M/R model by Bunday and Scraton [9].

Our objective in this chapter is to compare the results obtained using diffusion approximation with some existing exact results which have appeared in the literature such as the ones due to Ashcroft [4], Benson and Cox [6], Maritas and Xirokostas [44], and Bunday and Scraton [9].

5.1 The G/G/R Machine Repair Problem with No Spares

Following the diffusion approximation methodology presented in Chapter 3, we first obtain approximate formulas for the p.d.f. f(x) of the number of failed machines in the G/G/R MRP with no spares.

5.1.1 Steady-State Solutions for the G/G/R Machine Repair Problem

For a G/G/R queue, let

(i) λ^{-1} and σ_M^2 be the mean and variance respectively of the succession of uptimes for the operating machines;

(ii) μ^{-1} and σ_R^2 be the mean and variance respectively of the succession of repair times assumed to be the same at each repair station.

Note that in the diffusion approximation the distribution functions for the uptimes and repair times is not needed to compute f(x) but only their first two moments is required.

Let $C_M = \lambda^2\sigma_M^2$ and $C_R = \mu^2\sigma_R^2$, where C_M and C_R are the square coefficients of variation of the succession of the uptimes of the operating machines and the repair times, respectively.

The previous results for the warm standbys case in Chapter 3 may be used to obtain the results for the no spares as a special case. Using the results in equations 3.25b and 3.25c (Chapter 3) by setting S = 0 (N = M + S = M), we obtain the respective expressions for the p.d.f. $f_{13}(x)$, and $f_{14}(x)$ for the G/G/R MRP with no spares in terms of the dimensionless quantities $\theta_\lambda = \lambda/\mu$:

(5.1)
$$\begin{aligned}
f_{13}(x) &= \frac{K_{13}}{(M-x)\lambda C_M + x\mu C_R} \exp\left\{\int_0^x \frac{2[(M-y)\lambda - y\mu]}{(M-y)\lambda C_M + y\mu C_R}\,dy\right\} \\
&= K_{13}\,[(M-x)\theta_\lambda C_M + xC_R]^{-1}\left[\frac{(M-x)\theta_\lambda C_M + xC_R}{M\theta_\lambda C_M}\right]^{\beta_{13}} \\
&\quad * \exp\left\{\frac{2(\theta_\lambda+1)x}{\theta_\lambda C_M - C_R}\right\} \qquad \text{if } \theta_\lambda C_M - C_R \neq 0 \\
&= K_{13}\,(M\theta_\lambda C_M)^{-1} \exp\left\{\frac{2M\theta_\lambda x - (\theta_\lambda+1)x^2}{M\theta_\lambda C_M}\right\} \qquad \text{if } \theta_\lambda C_M - C_R = 0 \\
&\equiv K_{13}g_{13}(x), \qquad \text{for } 0 \le x \le R,
\end{aligned}$$

and

$$(5.2) \qquad f_{14}(x) = \frac{K_{14}}{(M-x)\lambda C_M + R\mu C_R} \exp\left\{ \int_R^x \frac{2[(M-y)\lambda - R\mu]}{(M-y)\lambda C_M + R\mu C_R} dy \right\}$$

$$= K_{14} \; [(M-x)\theta_\lambda C_M + RC_R]^{-1} \; \left[\frac{(M-x)\theta_\lambda C_M + RC_R}{(M-R)\theta_\lambda C_M + RC_R}\right]^{\beta_{14}}$$

$$* \exp\left\{\frac{2(x - R)}{C_M}\right\} \qquad \text{if } C_M > 0$$

$$= K_{14} \; (RC_R)^{-1} \exp\left\{\frac{2(M\theta_\lambda - R)(x-R) - \theta_\lambda(x^2 - R^2)}{RC_R}\right\} \quad \text{if } C_M = 0$$

$$= K_{14} g_{14}(x), \qquad \text{for } R < x \le M,$$

where

$$\beta_{13} = \frac{2M\theta_\lambda\left[1 + \dfrac{(\theta_\lambda + 1)C_M}{C_R - \theta_\lambda C_M}\right]}{C_R - \theta_\lambda C_M},$$

$$\beta_{14} = \frac{2R\left(1 + \dfrac{C_R}{C_M}\right)}{\theta_\lambda C_M}.$$

The unknown constants K_{13} and K_{14} are determined by two conditions. The first condition comes from the normalization criterion, namely,

$$(5.3) \qquad 1 = \int_0^M f(x)dx = K_{13} \int_0^R g_{13}(x)dx + K_{14} \int_R^M g_{14}(x)dx.$$

The second condition assumes continuity of $f(x)$ at $x = R$; i.e., $f_{13}(R) = f_{14}(R)$. Therefore we obtain

$$(5.4) \qquad K_{13}g_{13}(R) = K_{14}g_{14}(R).$$

The unknown constants K_{13}, and K_{14} are determined by solving equations 5.3 and 5.4 simultaneously.

5.1.2 Plot of the Steady-State Probability Density Functions f(x) for the M/M/R, M/G/R, G/M/R, and G/G/R Models

Once the unknown constants K_{13}, and K_{14} are obtained, the p.d.f. f(x) for the M/M/R, M/G/R, G/M/R, and G/G/R machine repair problems with no spares are plotted in Figures 5.1 and 5.2 for $\theta_\lambda = 0.4$, and for R = 3 and R = 6, respectively. These plots will have to be discretized in order to obtain estimates of the p.m.f. P_n. There are several ways using the discretization procedures which are suggested by Halachmi and Franta [25], Kobayashi [41], Kimura [35], Yao [73], and Gelenbe and Pujolle [20]. They are however representative of the behavior of P_n. For the G/G/R MRP with no spares, they are so far the only known plots available, although approximate.

5.1.3 System Characteristics for the G/G/R Machine Repair Problem

The approximate machine availability (MA_G) for the G/G/R MRP with no spares may be evaluated as

$$(5.5) \qquad MA_G = 1 - \{ E[X] / M \},$$

where E[X] denote the approximate expected number of failed machines given by

$$(5.6) \qquad E[X] = K_{13} \int_0^R x g_{13}(x) dx + K_{14} \int_R^M x g_{14}(x) dx.$$

The approximate operative utilization (OU_G) for the G/G/R MRP with no spares may be evaluated as

$$(5.7) \qquad OU_G = E[B]_G / R,$$

where $E[B]_G$ denote the approximate expected number of busy repairmen given by

$$(5.8) \qquad E[B]_G = R - \int_0^R (R - x) f_{13}(x) dx.$$

We vary the number of machines M from 8 to 24, and choose R = 3. Approximate results of the machine availability and the operative utilization for the G/G/R MRP with no spares are shown in Table 5.1 and Table 5.2 and are depicted in Figure 5.3 and Figure 5.4, respectively. Various values of θ_λ are considered. One sees from Table 5.1 and Table 5.2 that, as would be expected, for a given number of machines M the machine availability decreases with θ_λ while the operative utilization increases. On the other hand, for a given value of θ_λ, the machine availability decreases with M while the operative utilization increases.

Table 5.1 The approximate machine availability for the G/G/3 MRP with no spares ($C_M = 0.25$, $C_R = 0.25$).

	θ_λ							
M	0.15	0.2	0.25	0.3	0.4	0.5	0.6	0.8
8	0.869	0.833	0.800	0.768	0.708	0.646	0.583	0.463
10	0.869	0.833	0.798	0.762	0.680	0.585	0.498	0.375
12	0.869	0.831	0.790	0.740	0.615	0.499	0.417	0.313
14	0.868	0.825	0.769	0.692	0.535	0.429	0.357	0.268
16	0.866	0.812	0.726	0.623	0.469	0.375	0.312	0.234
18	0.862	0.784	0.663	0.555	0.417	0.333	0.278	0.208
20	0.852	0.737	0.600	0.500	0.375	0.300	0.250	0.188
22	0.834	0.680	0.545	0.455	0.341	0.273	0.227	0.170
24	0.804	0.625	0.500	0.417	0.312	0.250	0.208	0.156

Table 5.2 The approximate operative utilization for the G/G/3 MRP with no spares (C_M = 0.25, C_R = 0.25).

	θ_λ							
M	0.15	0.2	0.25	0.3	0.4	0.5	0.6	0.8
8	0.350	0.445	0.533	0.615	0.755	0.862	0.932	0.989
10	0.436	0.556	0.665	0.762	0.906	0.975	0.995	0.999
12	0.522	0.665	0.790	0.888	0.984	0.999	1.000	1.000
14	0.608	0.770	0.897	0.969	0.999	1.000	1.000	1.000
16	0.693	0.866	0.967	0.996	1.000	1.000	1.000	1.000
18	0.775	0.940	0.995	1.000	1.000	1.000	1.000	1.000
20	0.852	0.983	1.000	1.000	1.000	1.000	1.000	1.000
22	0.918	0.997	1.000	1.000	1.000	1.000	1.000	1.000
24	0.965	1.000	1.000	1.000	1.000	1.000	1.000	1.000

5.2 Comparative Analysis for the Machine Repair Problem with No Spares

The purpose of this section is to present specific numerical comparisons between the approximate results obtained using diffusion approximation and established exact results for the no spares case.

This section consists of the following three subsections:

(i) Comparative analysis between the exact and the diffusion approximation results to the M/M/R MRP.

(ii) Comparative analysis between some exact results and the corresponding diffusion approximation results to the MRP.

(iii) Comparative analysis for the system characteristics between some exact results and the corresponding diffusion approximation results to the MRP.

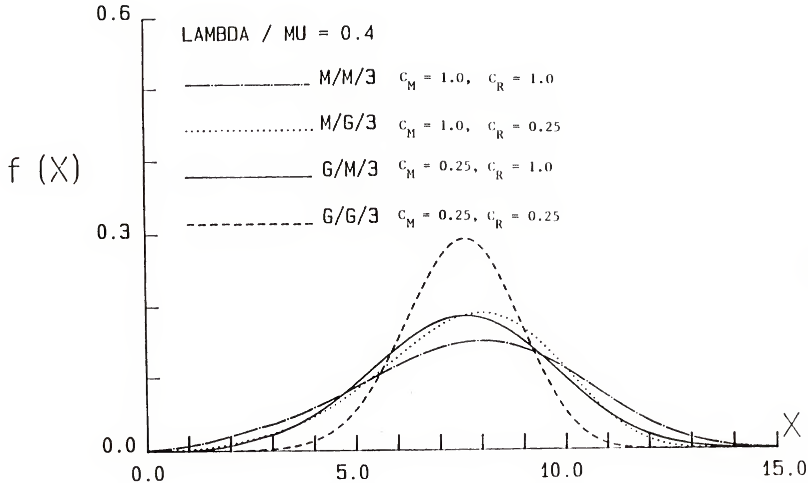


Figure 5.1 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

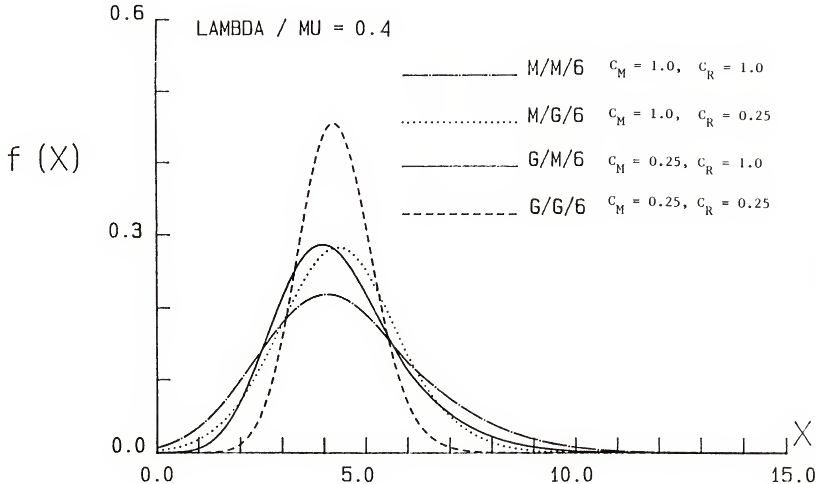


Figure 5.2 P.D.F. of the number of failed machines in the system in the steady-state using diffusion approximation.

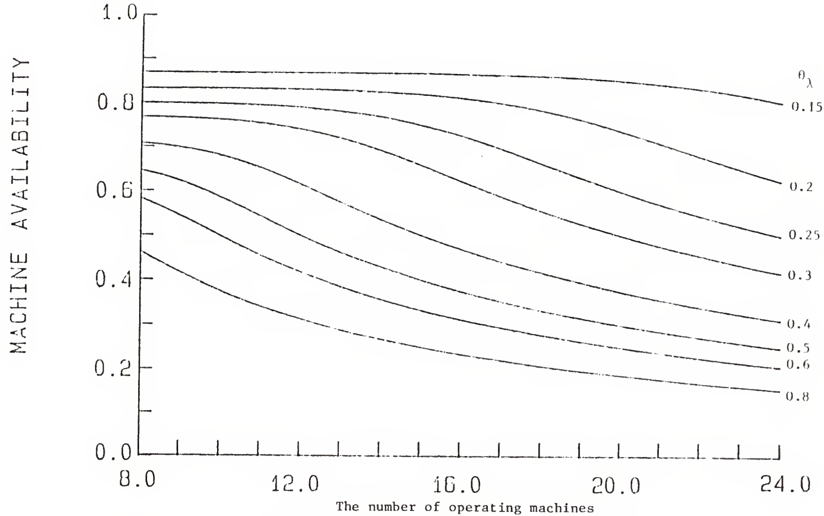


Figure 5.3 The approximate machine availability for the G/G/3 machine repair problem.

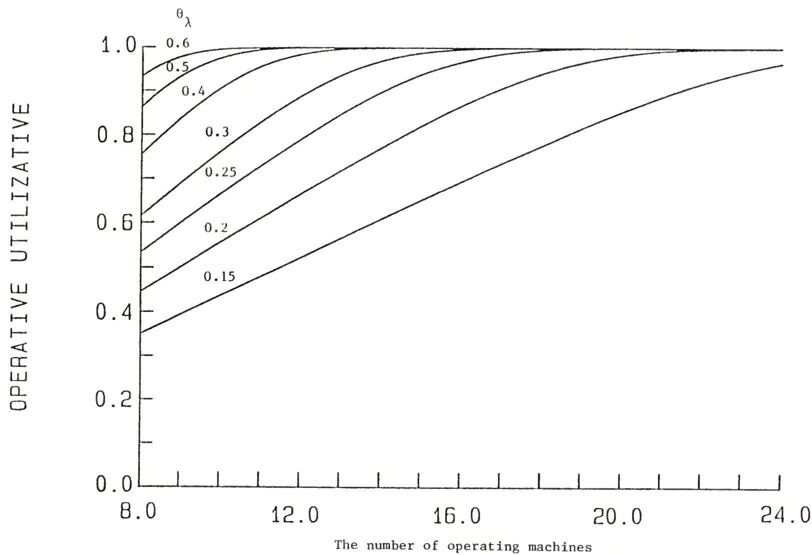


Figure 5.4 The approximate operative utilization for the G/G/3 machine repair problem.

5.2.1 Comparative Analysis Between the Exact and the Approximate Results to the M/M/R Machine Repair Problem

Here we perform a comparative analysis between the steady-state approximate p.d.f. f(x) obtained through diffusion approximation and the exact steady-state p.m.f. P_n obtained through birth and death equations for the M/M/R MRP.

We select the number of operating machines M = 15, and choose (i) the parameter $\theta_\lambda = 0.4$, varying the number of repairmen R, and (ii) the number of repairmen R = 3, varying the values of θ_λ. One observes from Figures 5.5 through 5.6 that the diffusion approximation provides very good results for the M/M/R MRP with no spares.

5.2.2 Comparative Analysis Between Some Exact Results and the Corresponding Diffusion Approximation Results to the Machine Repair Problem

Here we perform a comparative analysis between the approximate p.d.f. f(x) obtained through diffusion approximation and the exact p.m.f. P_n obtained from Benson and Cox [6] for the M/D/1 and the $M/E_K/1$ models.

We select the number of operating machines M = 10, and choose $\theta_\lambda = 0.2$, 0.3 and 0.4. Results for the M/D/1, the $M/E_2/1$, and the $M/E_{10}/1$ models are shown in Figures 5.7 through 5.9.

Next, we perform a comparative analysis for the system characteristics of the M/D/1, $M/E_k/1$, $M/E_k/R$, and G/M/R models between the approximate results obtained through diffusion approximation and the exact results obtained from Ashcroft [4] (M/D/1), Benson and Cox [6] ($M/E_k/1$), Maritas and Xirokostas [44] ($M/E_k/R$), and Bunday and Scraton [9] (G/M/R).

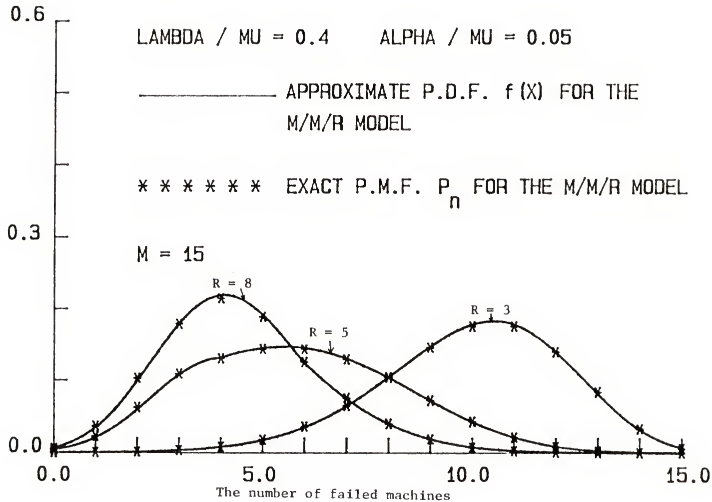


Figure 5.5 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

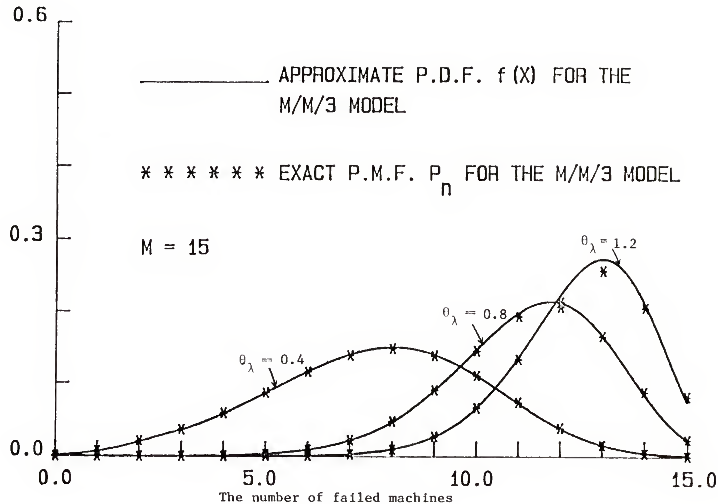


Figure 5.6 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

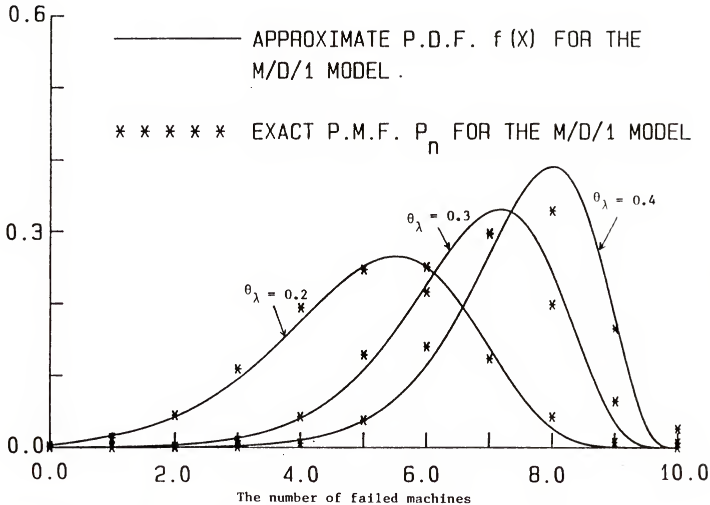


Figure 5.7 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

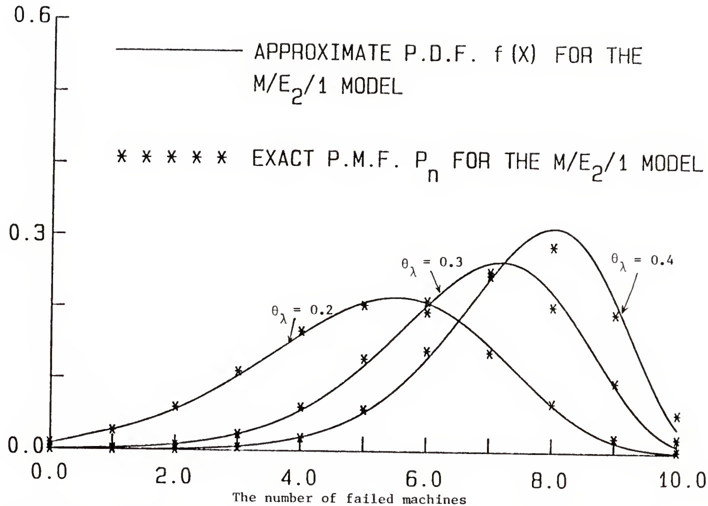


Figure 5.8 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

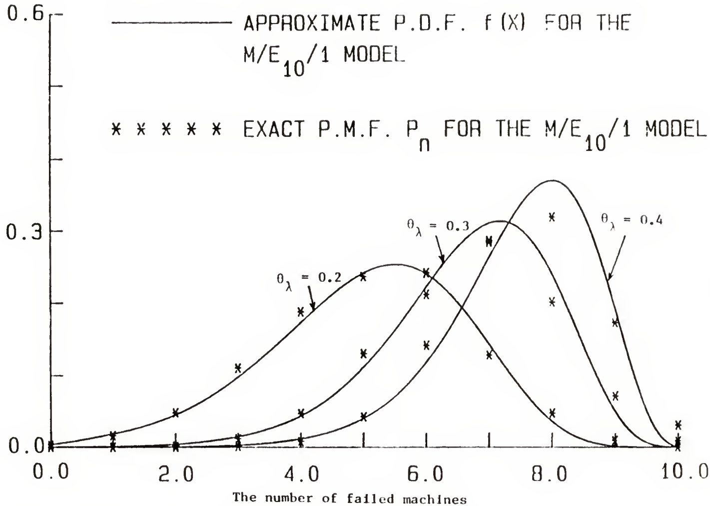


Figure 5.9 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

5.2.3 Comparative Analysis for the System Characteristics to the Machine Repair Problem

The objective of this subsection is to present specific computational results for the various measures of performance and to compare approximate results obtained using diffusion approximation with established exact results.

Here we perform a comparative analysis for the system characteristics to the MRP between the approximate results obtained through diffusion approximation, and the exact analytic results obtained from Ashcroft [4] (M/D/1), Benson and Cox [6] ($M/E_k/1$), Maritas and Xirokostas [44] ($M/E_k/R$), and Bunday and Scraton [9] (G/M/R). This comparison is performed for given specific numerical examples.

The subsection is divided into three parts which include the following:

(I) a comparative analysis for the expected number of failed machines in the M/D/1, $M/E_3/1$, and $M/E_4/1$ models between the diffusion approximation results and the exact analytic results obtained from Ashcroft [4], and Benson and Cox [6];

(II) a comparative analysis for the system characteristics in the $M/E_K/R$ model between the diffusion approximation results and the exact analytic results obtained from Maritas and Xirokostas [44];

(III) a comparative analysis for the system characteristics in the G/M/R MRP between the diffusion approximation results and the exact analytic results obtained from Bunday and Scraton [9]. In fact, these authors show that the G/M/R results are the same as the M/M/R results.

I. Comparative Analysis with Ashcroft's and Benson and Cox's Results

Here we first perform a comparative analysis for the expected number of failed machines in the M/D/1, $M/E_3/1$, and $M/E_4/1$ models between the approximate results and the exact analytic results obtained from Ashcroft [4], and Benson and Cox [6]. Results for the M/D/1, $M/E_3/1$, and $M/E_4/1$ models are shown in Table 5.3.

Next, we perform a comparative analysis for the machine availability in the $M/E_2/1$, $M/E_3/1$, and $M/E_4/1$ models between the approximate results and the exact results obtained from Benson and Cox [6]. Results for the $M/E_2/1$, $M/E_3/1$, and $M/E_4/1$ models are shown in Table 5.4.

II. Comparative Analysis with Maritas and Xirokostas's Results

Here we first perform a comparative analysis for the expected number of failed machines in the $M/E_3/2$ and $M/E_3/3$ models between the approximate results and the exact analytic results taken from Maritas and Xirokostas [44]. Results for the $M/E_3/2$ and $M/E_3/3$ models are presented in Table 5.5.

Next, we perform a comparative analysis for the machine availability and for the operative utilization in the $M/E_3/3$ model between the approximate results and the exact analytic results taken from Maritas and Xirokostas [44]. The comparison is done for $R = 3$, for different values of M ($4 \leq M \leq 20$) and for different values of θ_λ ($0.3 \leq \theta_\lambda \leq 1.2$).

Table 5.3 Comparison with Ashcroft's, and Benson and Cox's results for E[X].

(M/D/1 Model: $C_R = 0$)

	E[N]	E[X] (θ_λ = 0.3)		E[N]	E[X] (θ_λ = 0.6)	
M	Exact	Appro	% Error	Exact	Appro	% Error
4	1.333	1.381	3.601	2.369	2.410	1.731
6	2.766	2.819	1.916	4.334	4.339	0.115
8	4.670	4.685	0.321	6.333	6.334	0.016
10	6.667	6.668	0.015	8.333	8.333	0.000
12	8.667	8.667	0.000	10.333	10.333	0.000
14	10.667	10.667	0.000	12.333	12.333	0.000
16	12.667	12.667	0.000	14.333	14.333	0.000

(M/E_3/1 Model: $C_R = 1/3$)

	E[N]	E[X] (θ_λ = 0.3)		E[N]	E[X] (θ_λ = 0.6)	
M	Exact	Appro	% Error	Exact	Appro	% Error
4	1.390	1.481	6.547	2.395	2.418	0.960
6	2.818	2.878	2.129	4.335	4.318	0.392
8	4.680	4.699	0.406	6.333	6.309	0.379
10	6.667	6.668	0.015	8.333	8.309	0.288
12	8.667	8.665	0.023	10.333	10.309	0.232
14	10.667	10.665	0.019	12.333	12.309	0.195
16	12.667	12.665	0.016	14.333	14.309	0.167

(M/E_4/1 Model: $C_R = 1/4$)

	E[N]	E[X] (θ_λ = 0.3)		E[N]	E[X] (θ_λ = 0.6)	
M	Exact	Appro	% Error	Exact	Appro	% Error
4	1.377	1.458	5.882	2.389	2.420	1.298
6	2.806	2.864	2.067	4.334	4.328	0.138
8	4.677	4.696	0.406	6.333	6.320	0.205
10	6.667	6.669	0.030	8.333	8.320	0.156
12	8.667	8.666	0.011	10.333	10.320	0.126
14	10.667	10.666	0.009	12.333	12.320	0.105
16	12.667	12.666	0.008	14.333	14.320	0.091

Table 5.4 Comparison between the exact MA and the approximate MA_G

(A: $\theta_\lambda = 0.3$)

M	Exact $M/E_2/1$	Diff. Appr.	Exact $M/E_3/1$	Diff. Appr.	Exact $M/E_4/1$	Diff. Appr.
4	0.6463	0.6200	0.6524	0.6298	0.6556	0.6355
6	0.5263	0.5163	0.5303	0.5203	0.5324	0.5226
8	0.4141	0.4123	0.4150	0.4127	0.4153	0.4130
10	0.3332	0.3336	0.3333	0.3332	0.3333	0.3331
12	0.2778	0.2784	0.2778	0.2779	0.2778	0.2778
14	0.2381	0.2387	0.2381	0.2382	0.2381	0.2381
16	0.2083	0.2089	0.2083	0.2085	0.2083	0.2084

(B: $\theta_\lambda = 0.6$)

M	Exact $M/E_2/1$	Diff. Appr.	Exact $M/E_3/1$	Diff. Appr.	Exact $M/E_4/1$	Diff. Appr.
4	0.3984	0.3983	0.4014	0.3956	0.4029	0.3949
6	0.2773	0.2843	0.2775	0.2803	0.2776	0.2787
8	0.2083	0.2147	0.2083	0.2114	0.2083	0.2100
10	0.1667	0.1719	0.1667	0.1691	0.1667	0.1680
12	0.1389	0.1432	0.1389	0.1410	0.1389	0.1400
14	0.1190	0.1228	0.1190	0.1208	0.1190	0.1200
16	0.1042	0.1074	0.1042	0.1057	0.1042	0.1050

Maritas and Xirokostas [44] provide numerical results for the expected number of machines running for the models $M/E_3/2$ and $M/E_3/3$. We utilize their results to compare the approximate results. Exact values are taken from Maritas and Xirokostas's results. The approximate results are obtained using equations 5.1 and 5.2 to exhibit the numerical results in Table 5.6.

A comparison of the derived approximate results with some of the exact results in Tables 5.6, indicates that the relative percentage errors for the machine availability and for the operative utilization are very small for larger traffic intensity. However for low traffic

intensity levels and few machines, (e.g., $\theta_\lambda = 0.3$ and $M = 4$), the percentage error can be as high as 13% for the operative utilization.

III. Comparative Analysis with Bunday and Scraton's Results

First, we perform a comparative analysis for the expected number of failed machines in the G/M/3 model between the diffusion approximation results and the exact analytic results obtained from Bunday and Scraton [9]. Results for the G/M/3 model are presented in Table 5.7.

Next, we perform a comparative analysis for the machine availability and for the operative utilization in the G/M/R MRP between the diffusion approximation results and the exact analytic results obtained from Bunday and Scraton [9].

Bunday and Scraton show that for the no spare case, the distribution of the number of failed machines in the system for the G/M/R case is exactly the same as the M/M/R case. Thus, the exact values of P_n for the G/M/R case can be obtained from equations 2.7 in Chapter 2 by setting $\alpha = 0$ and $S = 0$. The approximate results are obtained using equations 5.1 and 5.2.

Table 5.5 Comparison with Maritas and Xirokostas's rsults for E[X].

($M/E_3/2$ Model: $C_R = 1/3$)

	E[*N*]	E[X]	($\theta_\lambda = 0.3$)	E[*N*]	E[X]	($\theta_\lambda = 0.6$)
M	Exact	Appro	% Error	Exact	Appro	% Error
4	0.966	1.064	10.145	1.649	1.621	1.698
6	1.603	1.639	2.246	2.948	2.964	0.543
8	2.494	2.504	0.401	4.704	4.717	0.276
10	3.767	3.786	0.504	6.669	6.670	0.015
12	5.439	5.460	0.386	8.667	8.665	0.023
14	7.349	7.359	0.136	10.667	10.665	0.019
16	9.333	9.337	0.043	12.667	12.665	0.016

($M/E_3/3$ Model: $C_R = 1/3$)

	E[*N*]	E[X]	($\theta_\lambda = 0.3$)	E[*N*]	E[X]	($\theta_\lambda = 0.6$)
M	Exact	Appro	% Error	Exact	Appro	% Error
4	0.926	1.044	12.743	1.521	1.570	3.221
6	1.443	1.484	2.841	2.491	2.394	3.894
8	2.102	1.980	5.804	3.747	3.557	5.071
10	2.949	2.614	11.360	5.266	5.151	2.200
12	3.966	3.491	11.977	7.054	7.027	0.383
14	5.141	4.703	8.520	9.006	9.003	0.033
16	6.525	6.262	4.031	11.000	11.000	0.000

($M/E_3/3$ Model: $C_R = 1/3$)

	E[*N*]	E[X]	($\theta_\lambda = 0.9$)	E[*N*]	E[X]	($\theta_\lambda = 1.2$)
M	Exact	Appro	% Error	Exact	Appro	% Error
4	1.939	1.945	0.309	2.249	2.226	1.023
6	3.233	3.123	3.402	3.752	3.669	2.212
8	4.832	4.762	1.449	5.540	5.515	0.451
10	6.689	6.677	0.179	7.502	7.494	0.107
12	8.668	8.666	0.023	9.500	9.493	0.074
14	10.667	10.665	0.019	11.500	11.493	0.061
16	12.667	12.665	0.016	13.500	13.493	0.052

Table 5.6 Comparison of Maritas and Xirokostas's results for the machine availability and the operative utilization ($M/E_3/3$ Model).

		θ_λ							
		0.3		0.6		0.9		1.2	
M		MA_G	OU_G	MA_G	OU_G	MA_G	OU_G	MA_G	OU_G
4	App.	0.739	0.348	0.608	0.521	0.514	0.640	0.444	0.723
	Exact	0.769	0.307	0.620	0.496	0.515	0.618	0.438	0.700
6	App.	0.753	0.490	0.601	0.739	0.479	0.871	0.389	0.934
	Exact	0.760	0.456	0.585	0.702	0.461	0.830	0.375	0.899
8	App.	0.752	0.629	0.555	0.896	0.405	0.973	0.311	0.992
	Exact	0.737	0.590	0.532	0.851	0.396	0.950	0.301	0.984
10	App.	0.739	0.756	0.485	0.972	0.332	0.996	0.251	0.999
	Exact	0.705	0.705	0.473	0.947	0.331	0.993	0.250	0.999
12	App.	0.709	0.861	0.414	0.995	0.278	1.000	0.209	1.000
	Exact	0.670	0.803	0.412	0.989	0.278	1.000	0.208	1.000
14	App.	0.664	0.934	0.357	0.999	0.238	1.000	0.179	1.000
	Exact	0.704	0.986	0.357	0.999	0.238	1.000	0.175	1.000
16	App.	0.609	0.976	0.312	1.000	0.208	1.000	0.157	1.000
	Exact	0.592	0.948	0.313	1.000	0.208	1.000	0.156	1.000
18	App.	0.551	0.993	0.278	1.000	0.185	1.000	0.139	1.000
	Exact	0.546	0.983	0.278	1.000	0.185	1.000	0.139	1.000
20	App.	0.499	0.998	0.250	1.000	0.167	1.000	0.125	1.000
	Exact	0.498	0.996	0.250	1.000	0.167	1.000	0.125	1.000

Table 5.7 Comparison with Bunday and Scraton's results for E[X].

(G/M/3 Model: $C_M = 0.25$)

	E[N]	E[X]	($\theta_\lambda = 0.3$)	E[N]	E[X]	($\theta_\lambda = 0.6$)
M	Exact	Appro	% Error	Exact	Appro	% Error
4	0.926	0.954	3.024	1.516	1.502	0.923
6	1.422	1.422	0.000	2.432	2.369	2.590
8	1.999	1.948	2.551	3.648	3.519	3.536
10	2.724	2.592	4.846	5.218	5.105	2.166
12	3.667	3.452	5.863	7.053	7.004	0.695
14	4.887	4.640	5.054	9.009	8.994	0.167
16	6.398	6.201	3.079	11.001	10.993	0.073

(G/M/R Model: $C_M = 0.25$, M = 10)

	E[N]	E[X]	(R = 3)	E[N]	E[X]	(R = 5)
θ_λ	Exact	Appro	% Error	Exact	Appro	% Error
0.2	1.804	1.759	2.494	1.670	1.678	0.479
0.4	3.646	3.473	4.745	2.906	2.884	0.757
0.6	5.218	5.105	2.166	3.919	3.856	1.608
0.8	6.298	6.242	0.889	4.778	4.683	1.988
1.0	7.012	6.967	0.642	5.496	5.394	1.856
1.2	7.504	7.450	0.720	6.084	5.991	1.529
1.4	7.858	7.792	0.840	6.561	6.480	1.235
1.6	8.126	8.046	0.984	6.948	6.876	1.036
1.8	8.333	8.242	1.092	7.264	7.196	0.936
2.0	8.500	8.399	1.188	7.525	7.456	0.917

The comparison is done for $R = 3$, for different values of M $(4 \leq M \leq 20)$ and for different values of θ_λ $(0.2 \leq \theta_\lambda \leq 0.8)$. We have selected arbitrary a value of $C_M = 0.5$ to exhibit the numerical results in Table 5.9.

A comparison of the derived approximate results with some of the exact results in Tables 5.8, indicates that the relative percentage errors for the machine availability and for the operative utilization are very small under larger traffic intensity. However for lower traffic intensity levels and few machines, (e.g., $\theta_\lambda = 0.2$, $\theta_\lambda = 0.3$, and $M = 4$), the percentage error can be as high as 13% for the operative utilization.

In Table 5.9, we compare the numerical results for $M = 10$, $R = 3$, and different values of C_M. The coefficient of variation of the uptimes of the operating machines vary between 0.1 and 1.25. One sees that the machine availability and the operative utilization are quite insensitive to C_M.

Finally, we perform a comparative analysis for the machine availability and the operative utilization for the M/D/1, $M/E_3/2$, $M/E_3/3$, and G/M/3 models together between the diffusion approximation results and the exact results obtained from Ashcroft [4], Maritas and Xirokostas [44], and Bunday and Scraton [9]. Numerical results of the comparison between the exact results and the approximate results are shown in Table 5.10.

Table 5.8 Comparison of Bunday and Scraton's results for the machine availability and the operative utilization (G/M/3 MOdel).

		θ_λ							
C_M = 0.5		0.2		0.4		0.6		0.8	
M		MA	OU	MA	OU	MA	OU	MA	OU
4	App.	0.806	0.259	0.700	0.397	0.619	0.502	0.554	0.584
	Exact	0.833	0.222	0.713	0.380	0.621	0.497	0.548	0.585
6	App.	0.818	0.359	0.699	0.572	0.600	0.722	0.518	0.822
	Exact	0.831	0.333	0.701	0.561	0.595	0.714	0.508	0.812
8	App.	0.821	0.460	0.682	0.733	0.554	0.886	0.451	0.954
	Exact	0.827	0.441	0.676	0.721	0.544	0.870	0.441	0.940
10	App.	0.819	0.561	0.645	0.863	0.486	0.970	0.375	0.994
	Exact	0.820	0.546	0.635	0.847	0.478	0.956	0.370	0.987
12	App.	0.811	0.659	0.591	0.947	0.416	0.995	0.315	1.000
	Exact	0.808	0.646	0.582	0.931	0.412	0.989	0.312	0.998
14	App.	0.798	0.751	0.528	0.985	0.358	1.000	0.270	1.000
	Exact	0.790	0.738	0.522	0.975	0.356	0.998	0.268	1.000
16	App.	0.777	0.833	0.468	0.997	0.313	1.000	0.236	1.000
	Exact	0.767	0.818	0.466	0.993	0.312	1.000	0.234	1.000
18	App.	0.748	0.900	0.417	1.000	0.278	1.000	0.210	1.000
	Exact	0.737	0.884	0.416	0.998	0.278	1.000	0.208	1.000
20	App.	0.711	0.948	0.375	1.000	0.251	1.000	0.189	1.000
	Exact	0.700	0.933	0.375	1.000	0.250	1.000	0.187	1.000

Table 5.9 Comparison of Bunday and Scraton's results for the machine availability and the operative utilization for different values C_M (G/M/3 Model).

M = 10		θ_λ							
		0.2		0.4		0.6		0.8	
C_M		MA_G	OU_G	MA_G	OU_G	MA_G	OU_G	MA_G	OU_G
	Exact	0.820	0.546	0.635	0.847	0.478	0.956	0.370	0.987
0.01	App.	0.827	0.551	0.660	0.880	0.493	0.985	0.376	0.999
0.04	App.	0.827	0.551	0.659	0.879	0.493	0.984	0.376	0.999
0.09	App.	0.826	0.551	0.657	0.876	0.492	0.983	0.376	0.998
0.16	App.	0.826	0.552	0.655	0.874	0.491	0.980	0.376	0.998
0.25	App.	0.824	0.553	0.653	0.870	0.489	0.978	0.376	0.997
0.36	App.	0.822	0.556	0.650	0.867	0.488	0.974	0.376	0.996
0.49	App.	0.819	0.561	0.646	0.863	0.486	0.970	0.375	0.994
0.64	App.	0.815	0.567	0.641	0.860	0.484	0.966	0.375	0.992
0.81	App.	0.811	0.575	0.636	0.857	0.481	0.961	0.374	0.989
1.00	App.	0.805	0.584	0.630	0.855	0.478	0.957	0.374	0.987
1.44	App.	0.793	0.605	0.617	0.854	0.471	0.949	0.371	0.980
2.25	App.	0.771	0.639	0.592	0.858	0.456	0.940	0.366	0.971

Table 5.10 Comparison between the exact results and the approximate results for the machine repair problem.

(A: $\theta_\lambda = 0.3$)

M		Exact M/D/1	Diff. Appr.	Exact $M/E_3/2$	Diff. Appr.	Exact $M/E_3/3$	Diff. Appr.	Exact G/M/3	Diff. Appr.
4	MA	0.667	0.655	0.758	0.734	0.769	0.739	0.769	0.762
	OU	0.800	0.867	0.455	0.518	0.307	0.348	0.307	0.317
6	MA	0.539	0.530	0.733	0.727	0.760	0.753	0.763	0.763
	OU	0.970	0.976	0.659	0.707	0.456	0.490	0.458	0.463
8	MA	0.416	0.414	0.688	0.687	0.737	0.752	0.750	0.757
	OU	0.999	0.997	0.826	0.854	0.590	0.629	0.600	0.607
10	MA	0.333	0.333	0.623	0.621	0.705	0.739	0.728	0.741
	OU	1.000	1.000	0.935	0.944	0.705	0.756	0.728	0.742
12	MA	0.278	0.278	0.547	0.545	0.670	0.709	0.694	0.712
	OU	1.000	1.000	0.984	0.985	0.803	0.861	0.833	0.855
14	MA	0.238	0.238	0.475	0.474	0.633	0.664	0.651	0.669
	OU	1.000	1.000	0.998	0.997	0.886	0.934	0.911	0.936
16	MA	0.208	0.208	0.417	0.416	0.592	0.609	0.600	0.612
	OU	1.000	1.000	1.000	1.000	0.948	0.976	0.960	0.980

(B: $\theta_\lambda = 0.6$)

M		Exact M/D/1	Diff. Appr.	Exact $M/E_3/2$	Diff. Appr.	Exact $M/E_3/3$	Diff. Appr.	Exact G/M/3	Diff. Appr.
4	MA	0.408	0.397	0.588	0.595	0.620	0.608	0.621	0.624
	OU	0.978	0.978	0.705	0.751	0.496	0.521	0.497	0.496
6	MA	0.278	0.277	0.509	0.506	0.585	0.601	0.595	0.605
	OU	1.000	0.999	0.916	0.927	0.702	0.739	0.714	0.724
8	MA	0.208	0.208	0.412	0.410	0.532	0.555	0.544	0.560
	OU	1.000	1.000	0.989	0.988	0.851	0.896	0.870	0.895
10	MA	0.167	0.167	0.333	0.333	0.473	0.485	0.478	0.489
	OU	1.000	1.000	0.999	0.999	0.947	0.972	0.956	0.978
12	MA	0.139	0.139	0.278	0.278	0.412	0.414	0.412	0.416
	OU	1.000	1.000	1.000	1.000	0.989	0.995	0.989	0.998
14	MA	0.119	0.119	0.238	0.238	0.357	0.357	0.356	0.358
	OU	1.000	1.000	1.000	1.000	0.999	0.999	0.998	1.000
16	MA	0.104	0.104	0.208	0.208	0.312	0.312	0.312	0.313
	OU	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Through comparative analysis of I, II, and III for the MRP with no spares, we show that the use of diffusion approximation is accurate enough for practical purposes. Diffusion approximation provides a useful analytic tool to obtain systems characteristics of complex machine repair problems and thus is helpful in analyzing real life situations which otherwise would have been too complex to solve analytically.

CHAPTER 6

COMPARATIVE ANALYSIS FOR THE MACHINE REPAIR PROBLEM WITH WARM STANDBYS

Our primary objective in this chapter is to present specific numerical comparisons between exact results, approximate results obtained using diffusion approximation, and simulation results.

This chapter is divided into three sections which include the following:

(I) A comparative analysis between exact results and diffusion approximation results to the M/M/R MRP with warm standbys.

(II) A comparative analysis for the system characteristics between diffusion approximation results and simulation results to the M/M/R MRP and $E_2/E_2/R$ machine repair problems with warm standbys.

(III) A sensitivity analysis for the system characteristics to the G/G/R MRP with warm standbys when varying the square coefficients of variation C_M, C_S, and C_R.

6.1 Comparative Analysis Between Exact Results and Approximate Results to the M/M/R Machine Repair Problem with Warm Standbys

The previous results obtained for the exact p.m.f. P_n in Chapter 2 and the approximate p.d.f. f(x) in Chapter 3 are utilized to provide a comparative analysis for the system characteristics to the M/M/R MRP with warm standbys. This section is divided into two subsections. In

the first subsection, we perform a comparative analysis between the exact p.m.f. P_n and the approximate p.d.f. f(x) to the M/M/R MRP with warm standbys. In the second subsection, we perform a comparative analysis for the system characteristics between the exact analytic results and the approximate results to the M/M/R MRP with warm standbys.

6.1.1 Comparative Analysis Between the Exact P.M.F. P_n and the Approximate P.D.F. f(x)

Here we perform a comparative analysis between the exact p.m.f. P_n and the approximate p.d.f. f(x) to the M/M/R MRP with warm standbys. The exact values for p.m.f. P_n to the M/M/R MRP with warm standbys are shown in Tables 2.1 and 2.2 in Chapter 2. We select the number of operating machines M = 10, the number of spares S = 5, and choose (i) the parameter $\theta_\lambda = 0.4$, while varying the number of repairmen R, and (ii) the number of repairmen R = 3, while varying the values of θ_λ. One observes from Figures 6.1 through 6.2 that the diffusion approximation provides very good results for the M/M/R MRP with warm standbys.

6.1.2 Comparative Analysis for the System Characteristics Between Exact Results and Approximate Results

Next, we perform a comparative analysis for the expected number of failed machines in the system for the M/M/R MRP with warm standbys between the exact analytic results, and the approximate results.

Let

EXA ≡ the exact expected number of failed machines for the exact model,

DIF_d ≡ the approximate expected number of failed machines for the diffusion model,

$DIF_r \equiv$ the approximate expected number of failed machines using heavy traffic condition and renewal theory.

The values of EXA are taken from Tables 2.3 and 2.4 in Chapter 2, the values of DIF_d are taken from Tables 2.7 and 2.8 in Chapter 2, and the values of DIF_r are taken from Tables 3.9 and 3.10 in Chapter 3. Table 6.1 provides the comparisons between the exact analytic results (EXA), the approximate results (DIF_d), and the approximate results (DIF_r) using the assumption of heavy traffic conditions and asymptotic results from renewal theory, as a function of θ_λ, for $\theta_\alpha = 0.05$, and for various number of repairmen. The exact mean and the approximate mean number of failed machines in the M/M/R MRP with spares are depicted in Figure 6.3, as a function of θ_λ, for $\theta_\alpha = 0.05$, and for various number of repairmen.

One sees from Table 6.1 that the results (DIF_r) obtained using heavy traffic conditions and renewal theory are very close to the exact results, thus showing that the use of diffusion approximation is accurate enough for practical purposes.

Finally, we show that diffusion approximation provides a useful analytic tool to calculate system characteristics, such as for example, machine availability and operative utilization, for the complex machine repair problems. To this end, we perform a comparative analysis for the machine availability and for the operative utilization in the M/M/R MRP with warm standbys between the exact analytic results and the diffusion approximation results.

Table 6.1 Comparisons of exact and approximate results for the M/M/R MRP with warm standbys (M=10, S=5, N = 15, θ_{α} = 0.05).

θ_{λ}		R = 1	3	5	6	8	10
	EXA	9.9805	2.7956	2.1735	2.1461	2.1396	2.1394
0.2	DIF_d	9.9514	2.9587	2.3687	2.3440	2.3388	2.3388
	DIF_r	9.9620	2.9240	2.2887	2.2572	2.2480	2.2476
	EXA	12.4999	7.3972	4.4707	4.1074	3.9485	3.9374
0.4	DIF_d	12.3329	7.4113	4.5065	4.1566	4.0119	4.0032
	DIF_r	12.4062	7.4099	4.5093	4.1454	3.9807	3.9670
	EXA	13.3333	9.9812	6.9250	6.0775	5.5016	5.4272
0.6	DIF_d	13.0595	9.9500	6.9227	6.0775	5.5206	5.4555
	DIF_r	13.1823	9.9620	6.9336	6.0895	5.5123	5.4333
	EXA	13.7500	11.2475	8.7758	7.7807	6.7895	6.5863
0.8	DIF_d	13.4031	11.1715	8.7619	7.7696	6.7865	6.5980
	DIF_r	13.5626	11.2041	8.7703	7.7817	6.7938	6.5871
	EXA	14.0000	11.9996	10.0010	9.0761	7.8643	7.5032
1.0	DIF_d	13.6019	11.8769	9.9680	9.0566	7.8489	7.5027
	DIF_r	13.7895	11.9304	9.9827	9.0685	7.8649	7.5035
	EXA	14.1667	12.4999	10.8326	10.0223	8.7602	8.2553
1.2	DIF_d	13.7310	12.3329	10.7748	9.9880	8.7373	8.2444
	DIF_r	13.9410	12.4062	10.7999	10.0045	8.7568	8.2551
	EXA	14.2857	12.8571	11.4282	10.7214	9.5014	8.8894
1.4	DIF_d	13.8215	12.6500	11.3428	10.6675	9.4711	8.8696
	DIFr	14.0500	12.7415	11.3800	10.6918	9.4924	8.8876
	EXA	14.3750	13.1250	11.8748	11.2525	10.1119	9.4332
1.6	DIF_d	13.8885	12.8824	11.7614	11.1763	10.0717	9.4050
	DIF_r	14.1326	12.9903	11.8109	11.2100	10.0956	9.4282
	EXA	14.4444	13.3333	12.2221	11.6676	10.6153	9.9038
1.8	DIF_d	13.9399	13.0595	12.0813	11.5682	10.5626	9.8671
	DIF_r	14.1975	13.1823	12.1429	11.6118	10.5903	9.8942
	EXA	14.5000	13.5000	12.5000	12.0004	11.0328	10.3135
2.0	DIF_d	13.9808	13.1987	12.3330	11.8778	10.9654	10.2676
	DIF_r	14.2500	13.3349	12.4063	11.9315	10.9883	10.2981

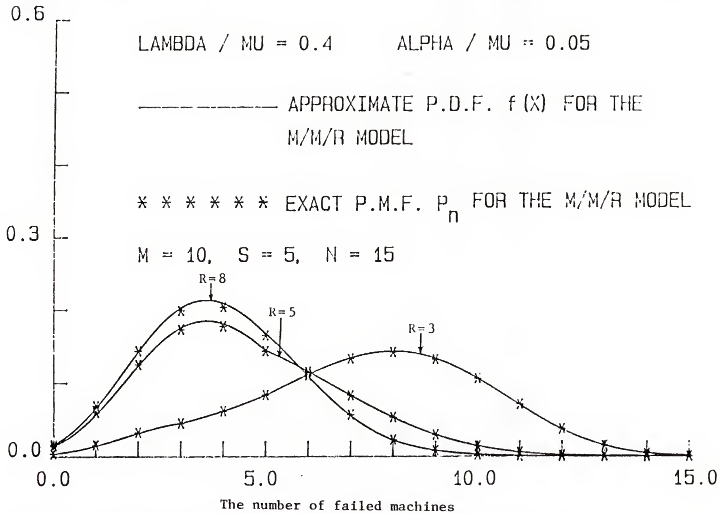
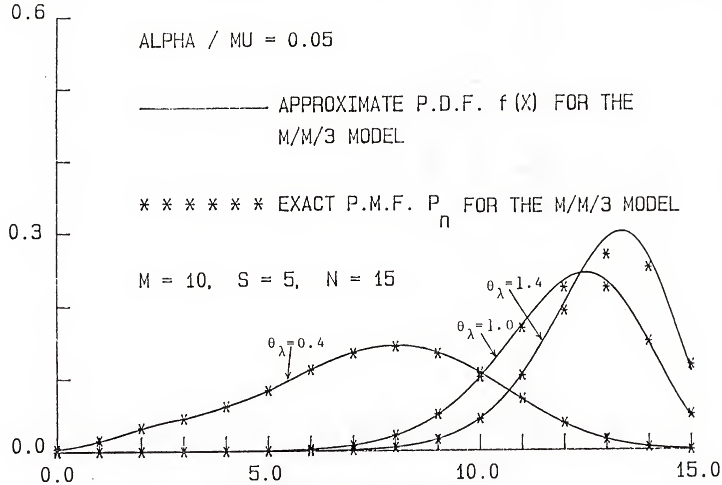


Figure 6.1 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.



The number of failed machines

Figure 6.2 Comparison between exact P.M.F. and approximate P.D.F. for the number of failed machines in the system in the steady-state.

This analysis is carried out for the two following cases:

Case (1) Here we select the number of operating machines M = 10, the number of repairmen R = 6, the parameter $\theta_\alpha = 0.05$, while varying the number of spare machines S ($1 \le S \le 11$) and the parameter θ_λ ($0.2 \le \theta_\lambda \le 0.8$).

The exact and approximate results of the machine availability and the operative utilization for the M/M/R MRP with spares are shown in Table 6.2. Comparing the two results for the machine availability and for the operative utilization indicate that when one selects the lower value $\theta_\lambda = 0.2$, the relative percentage errors are small (0-7%). As may be expected the approximations get better as one gets closer to higher value of θ_λ. This suggests that the diffusion approximation approach provides very good approximation.

Case (2) Here we select the number of operating machines M = 10, the number of spare machines S = 5, the parameter $\theta_\alpha = 0.05$ while varying the number of repairmen R ($3 \le R \le 8$) and the parameter θ_λ ($0.2 \le \theta_\lambda \le 2.0$).

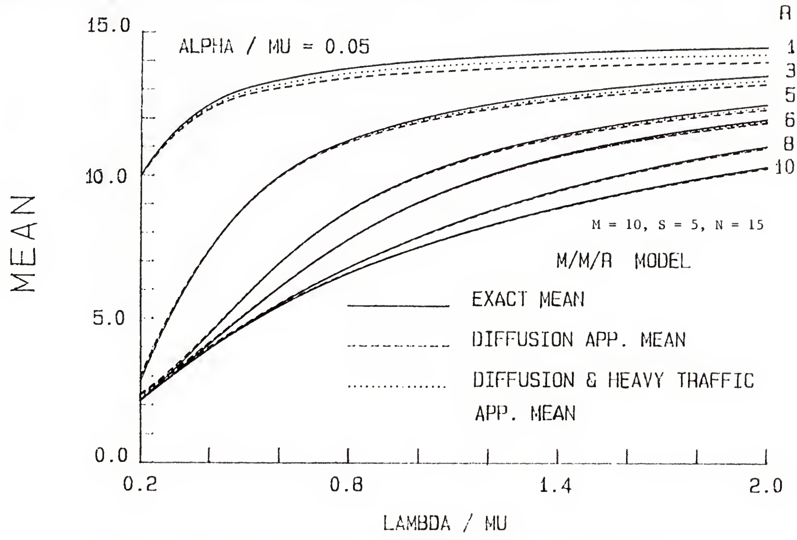


Figure 6.3 The exact and approximate expected number of failed machines in the M/M/R machine repair problem.

Table 6.2 Comparisons of diffusion approximation and exact results for machine availability and operative utilization to the M/M/R MRP with warm standbys (M = 10, R = 6, θ_α = 0.05).

		θ_λ							
		0.2		0.4		0.6		0.8	
S		MA	OU	MA	OU	MA	OU	MA	OU
1	App.	0.824	0.322	0.709	0.528	0.616	0.680	0.537	0.788
	Exact	0.835	0.303	0.713	0.521	0.617	0.679	0.538	0.789
2	App.	0.829	0.341	0.711	0.568	0.611	0.733	0.526	0.841
	Exact	0.839	0.322	0.714	0.563	0.613	0.732	0.527	0.842
3	App.	0.836	0.355	0.713	0.603	0.606	0.779	0.513	0.885
	Exact	0.844	0.336	0.716	0.599	0.607	0.779	0.513	0.886
4	App.	0.843	0.366	0.718	0.631	0.600	0.817	0.498	0.918
	Exact	0.851	0.347	0.721	0.627	0.601	0.818	0.498	0.919
5	App.	0.850	0.374	0.724	0.652	0.594	0.848	0.481	0.943
	Exact	0.857	0.356	0.726	0.648	0.595	0.848	0.481	0.943
6	App.	0.856	0.382	0.730	0.668	0.588	0.870	0.464	0.959
	Exact	0.863	0.365	0.733	0.664	0.589	0.870	0.464	0.960
7	App.	0.861	0.390	0.737	0.680	0.582	0.888	0.447	0.971
	Exact	0.868	0.373	0.739	0.676	0.582	0.888	0.447	0.971
8	App.	0.866	0.397	0.744	0.691	0.575	0.903	0.430	0.979
	Exact	0.872	0.381	0.746	0.687	0.576	0.903	0.430	0.980
9	App.	0.871	0.405	0.750	0.701	0.569	0.915	0.413	0.985
	Exact	0.876	0.389	0.752	0.697	0.569	0.915	0.413	0.985
10	App.	0.875	0.412	0.757	0.710	0.562	0.926	0.397	0.989
	Exact	0.880	0.397	0.759	0.706	0.563	0.926	0.397	0.990
11	App.	0.879	0.420	0.763	0.718	0.555	0.935	0.381	0.992
	Exact	0.883	0.405	0.764	0.714	0.556	0.935	0.381	0.993

The exact and approximate results of machine availability and operative utilization for the M/M/R MRP with spares are shown in Table 6.3. Comparing the derived approximate results with some of the exact results in Table 6.3, indicates that the relative percentage errors for the machine availability and for the operative utilization are small. However for $\theta_\lambda = 2.0$, and R = 3, the percentage error can be as high as 10% for the machine availability.

Table 6.3 may be used for specifying the minimum number of repairmen to achieve a given level of operative utilization. For example, when M = 10, S = 5 and $\theta_\lambda = 0.8$, then, in order that the repairmen remain busy at least 94% of the time, 6 repairmen would be required. Under these conditions, the machine availability is 0.481 or the expected percentage of machines running is approximately 50%. In any real life situation, it is desirable to keep a high level of machine availability and operative utilization. The right compromise must rely on a well defined objective function such as cost or profit function. However, even without such objective functions, Table 6.3 provides a reasonable amount of information to the decision-maker. For example, assuming a $\theta_\lambda = 0.4$, then if the manager specifies that the repairmen must be busy for at least 75% of the time and the machines must be working for at least 65% of the time, then the number of repairmen to be assigned is R = 5.

Table 6.3 Comparisons of diffusion approximation and exact results for machine availability and operative utilization to the M/M/R MRP with war standbys ($M = 10$, $S = 5$, $\theta_\alpha = 0.05$).

		R							
		3		5		6		8	
θ_λ		MA	OU	MA	OU	MA	OU	MA	OU
0.2	App.	0.805	0.720	0.847	0.448	0.850	0.374	0.850	0.281
	Exact	0.814	0.692	0.855	0.427	0.857	0.356	0.857	0.267
0.3	App.	0.653	0.904	0.778	0.617	0.788	0.519	0.791	0.390
	Exact	0.657	0.896	0.783	0.606	0.792	0.509	0.795	0.383
0.4	App.	0.506	0.976	0.699	0.762	0.724	0.652	0.735	0.495
	Exact	0.507	0.975	0.702	0.757	0.726	0.648	0.737	0.492
0.5	App.	0.404	0.994	0.616	0.867	0.658	0.764	0.681	0.590
	Exact	0.403	0.994	0.617	0.866	0.660	0.762	0.683	0.589
0.6	App.	0.336	0.999	0.538	0.931	0.594	0.848	0.633	0.673
	Exact	0.335	0.999	0.538	0.931	0.595	0.848	0.633	0.673
0.7	App.	0.288	1.000	0.470	0.966	0.534	0.905	0.588	0.742
	Exact	0.286	1.000	0.471	0.966	0.535	0.906	0.588	0.743
0.8	App.	0.253	1.000	0.415	0.983	0.481	0.943	0.547	0.799
	Exact	0.250	1.000	0.415	0.984	0.481	0.943	0.547	0.800
0.9	App.	0.226	1.000	0.371	0.991	0.435	0.965	0.510	0.844
	Exact	0.222	1.000	0.370	0.992	0.435	0.966	0.510	0.845
1.0	App.	0.205	1.000	0.334	0.995	0.395	0.979	0.476	0.880
	Exact	0.200	1.000	0.333	0.996	0.395	0.980	0.476	0.881
1.2	App.	0.173	1.000	0.280	0.999	0.333	0.992	0.416	0.930
	Exact	0.167	1.000	0.278	0.999	0.332	0.993	0.416	0.931
1.6	App.	0.134	1.000	0.213	1.000	0.253	0.999	0.327	0.975
	Exact	0.125	1.000	0.208	1.000	0.250	0.999	0.326	0.976
2.0	App.	0.111	1.000	0.173	1.000	0.205	1.000	0.267	1.000
	Exact	0.100	1.000	0.167	1.000	0.200	1.000	0.264	0.991

6.2 Comparative Analysis for the System Characteristics Between Approximate Results and Simulation Results to the M/M/R and $E_2/E_2/R$ Machine Repair Problems with Warm Standbys

Exact and tractable steady-state solutions for the G/G/R MRP or the G/G/R system are yet unknown. However, several authors have used the diffusion approximation to investigate (i) the G/G/R MRP with no spares (Wang and Sivazlian [67]), (ii) the G/G/R MRP with cold standbys (Haryono and Sivazlian [27]), (iii) the G/G/R MRP with warm standbys (Sivazlian and Wang [56]), (iv) the GI/G/1 system (Heyman [29]) and (v) the G/G/R system (Halachmi and Franta [25]). A comparison between the results obtained from diffusion approximation and the simulation results is made by Heyman [29], Halachmi and Franta [25], and Haryono and Sivazlian [27]. Simulation results are provided for the MRP (i) with no spares by Law and Kelton [43], and (ii) with cold standbys by Schriber [53, 54] and Haryono and Sivazlian [27].

In Chapter 4, we carried out a comparative analysis of the G/G/R machine repair problem with no spares. In this chapter, we investigate some aspects of the G/G/R MRP with warm standbys by employing simulation techniques and comparing them with the analytic results.

6.2.1 Introduction

The purpose of Section 6.2 is twofold. The first one is to show how theoretical results can be used to arrive at intelligent decisions in analyzing simulation results in the context of determining cut-off times for termination of the transients. The second one is to show how simulation can be used to solve problems for which theory does not provide accessible mathematical techniques for obtaining analytic results. The illustrations provided show on the one hand how theory can

lend support to simulation and on the other hand how simulation can lend support to theory. The illustration is drawn from the G/G/R MRP with warm standbys.

The first problem, namely, the determination of the cut-off time at which the simulation run reaches, for all practical purposes, steady-state, is performed by solving the transient M/M/R MRP with warm standbys. The time-dependent solution for two extreme initial conditions provide useful indication of the epochs at which the simulation transient should be discarded and steady-state should be measured. The results are used to analyze the simulation results for a G/G/R MRP discussed next.

The second problem relates to the order of selection of available spares for replacing failed machines in the G/G/R MRP. The inclusion of a priority rule for such a selection mode is extremely difficult if not impossible to handle analytically. Thus, theory cannot at this stage provide insight into how such selection mode affects the behavior of the stochastic process underlying the G/G/R MRP. On the other hand simulation becomes a useful tool to gain such insight even at a rudimentary level. The techniques for the cut-off time of transients are used in this problem.

Prior to performing the analysis, we digress on two important considerations related to the simulation. The first one relates to the "cut-off" time at which the simulation run, for all practical purposes, reaches steady-state. The second one relates to the order of selection of available spares for replacing failed operating machines. The simulation results of the spare selection order study are compared to

the exact and diffusion approximation results of previous work (Sivazlian and Wang [55, 56]). This includes:

(i) A comparative analysis for the system characteristics between exact results, diffusion approximation results, and simulation results to the M/M/R MRP with warm standbys.

(ii) A comparative analysis for the system characteristics between diffusion approximation results and simulation results to the $E_2/E_2/R$ MRP with warm standbys.

6.2.2 Selection of Simulation "Cut-Off" Time for Steady-State

Simulation is a powerful and practical tool which can be used to analyze and investigate many complex queueing systems. Simulation methods are usually easier to utilize than analytic methods for complex queueing problems. Very often, we may require many specific assumptions to make analytic models mathematically tractable, but we have no such restrictions for simulation models. Simulation on the other hand has several drawbacks. It is well known, for example, that simulation may become expensive in terms of manpower and computer time. It obviously cannot provide the quantitative relationships that are desirable from theory. At a more practical level, it is often difficult to determine from a simulation run when, for all practical purposes, steady state has been reached. Steady state is assumed to have been reached when successive computation values no longer vary significantly. If the system is relatively close to steady state, the analyst must decide how to reduce or eliminate the bias caused by the transient period. The following methodology is discussed to provide a partial answer to the above problem, when making a simulation analysis of the G/G/R MRP.

We use Laplace transform to obtain the closed-form transient (time-dependent) solutions $P(n;t)$ for the M/M/R MRP with warm standbys, where $P(n;t)$ denotes the probability of n failed machines in the system at time t. No analysis or solution methodology is provided in the present work. The reader is referred to Wang and Sivazlian [68] for a more detailed analysis and solution methodology.

It should be noted that the transient solutions are independent of the initial conditions for large values of t. There is no particular criterion or explicit expressions to illustrate the length of warm-up period. In the study of signals, within 2% of the steady-state is considered the standard setting time. According to the proposed methodology, we may evaluate the "cut-off" time of the simulation at approximately the time when the p.m.f. $P(n;t)$ for the two extreme cases described below meet. The two extreme cases arise from two extreme initial conditions. These are:

(1) Initial condition I:

$P_I(0;0) = 1,$

$P_I(n;0) = 0, \quad \text{for } n = 1, 2, \ldots, N;$

(2) Initial condition II:

$P_{II}(n;0) = 0, \quad \text{for } n = 0, 1, \ldots, N - 1,$

$P_{II}(N;0) = 1.$

These conditions correspond respectively to the situations when at $t = 0$ no machines are in the failed state and all machines are in the failed state.

For example, if we set the number of operating machines to $M = 10$, the number of warm standbys to $S = 5$, and choose $\alpha = 0.05$, $\lambda = 0.2$, and

$\mu = 1.0$, we see from the plot of $\sum_{n=R}^{N} P(n;t)$, which is the probability that all repairmen are busy, that the warm-up period may be taken as approximately $t = 15$ (Figure 6.4). This gives a reasonable criterion to determine the warm-up period which may be used in simulation work.

A more suitable approach consists of comparing the plots of the probability mass function $P(n;t)$ for the two extreme initial conditions for different values of t and selecting that time when these plots coincide according to a preselected criterion. The criterion used here is $| P_I(n;t) - P_{II}(n;t) | \leq \epsilon$, for all values of n, where ϵ is a small quantity. The cut-off times for the M/M/R MRP with warm standbys for a specific example used throughout this paper are shown in Table 6.4 ($\epsilon = 10^{-6}$). Figures 6.5 through 6.7 depict $P(n;1)$, $P(n;4)$, and $P(n;7)$, where the dashed curves are obtained for the <u>initial condition I</u>, and the dotted curves are obtained for the <u>initial condition II</u>.

One sees from Table 6.4 that for a given value of λ, the cut-off time decreases as R increases. On the other hand, for a given value of R, the cut-off time in general decreases as λ increases.

The closed-form transient solution for the G/G/R MRP with warm standbys is as yet not available. However, a cut-off time for the G/G/R MRP with warm standbys may be taken to be approximately the one obtained from the transient solution of the M/M/R case. This would be a reasonable approach to resolve a rather complex situation.

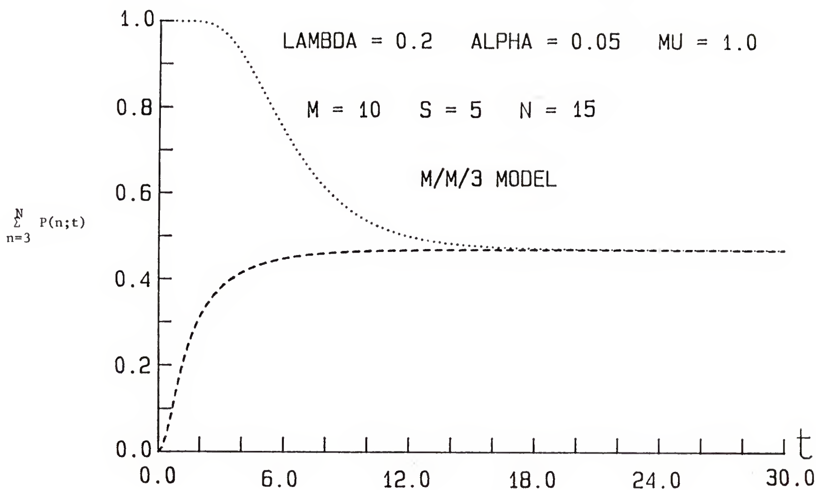


Figure 6.4 The transient form of the function $\sum_{n=3}^{N} P(n;t)$ with Initial Condition I (dashed curve), and Initial Condition II (dotted curve).

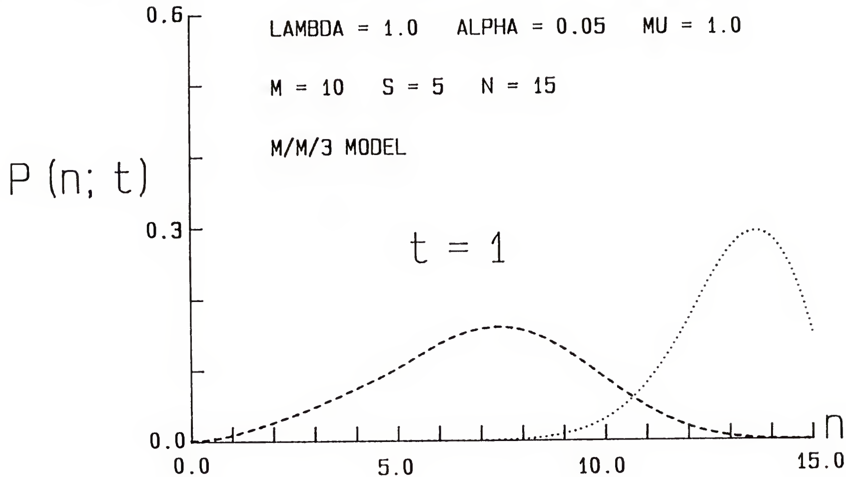


Figure 6.5 The probability mass function P(n;1) with Initial Condition I (dashed curve), and Initial Condition II (dotted curve).

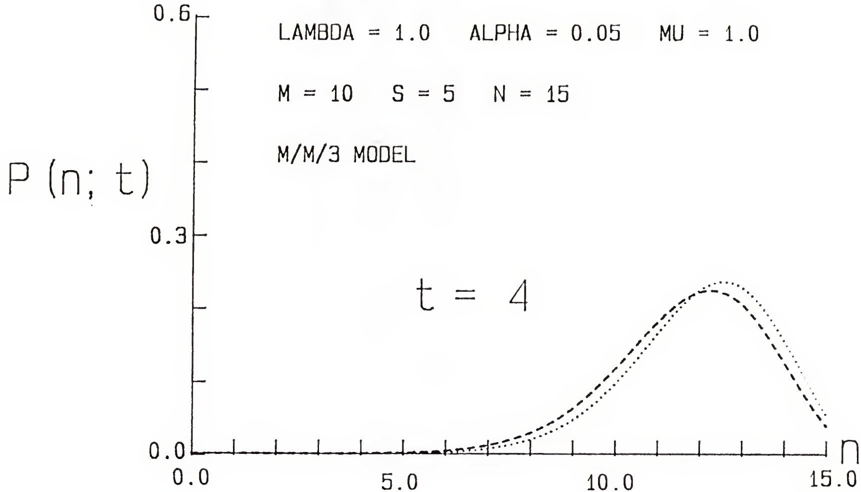


Figure 6.6 The probability mass function P(n;4) with Initial Condition I (dashed curve), and Initial Condition II (dotted curve).

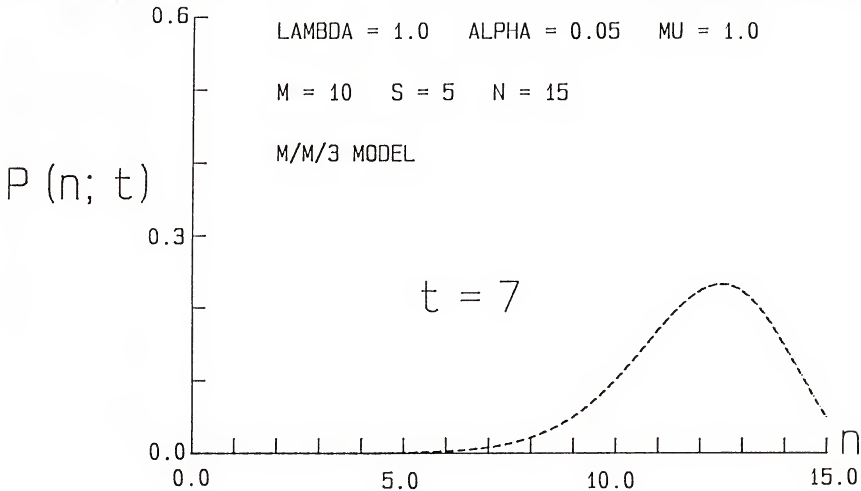


Figure 6.7 The probability mass function P(n;7) with Initial Condition I (dashed curve), and Initial Condition II (dotted curve).

Table 6.4 The cut-off times for termination of the transient to the M/M/R machine repair problem with warm standbys ($M = 10$, $S = 5$, $\alpha = 0.05$, $\mu = 1.0$).

	λ									
R	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
1	60	47	34	25	21	18	15	14	13	11
3	43	44	31	25	20	17	16	13	12	11
5	19	24	22	22	18	16	14	13	11	11
6	19	19	20	20	18	16	14	12	11	10
8	19	17	15	13	13	12	12	11	10	10
10	17	16	14	12	11	10	10	9	8	8

6.2.3 Order of Selection of Available Spares for Replacing Failed Operating Machines

It was mentioned in Section 1.1 (Problem Statement) that when an operating machine or a spare machine breaks down, it is immediately sent to the repair facility for repair in the order of their breakdown, and they are treated as identical machines to be repaired. However, special consideration must be given for choosing the available spares for replacing failed operating machines. The order of selection of the available spares becomes thus a factor which cannot be neglected. If spares are considered to age during the standby period, whether we choose a recently repaired spare or an aged one to join the operating pool of machines has an effect on the system behavior. This factor does not arise in the M/M/R MRP due to the memoryless property of the exponential distribution, that is, if the spares do not age. It arises however in the general G/G/R MRP.

In previous studies, we neglected to consider the order of selection of available spares as a significant factor. The solutions for the steady-state problem incorporating a selection mode would have

been very difficult, if not impossible, to obtain analytically. Even a heuristic argument may not overcome these difficulties for the G/G/R MRP with spares using the diffusion approximation approach. Therefore, one has to resort to simulation methods to provide the necessary experiment to evaluate and compare different selection modes for choosing spares to replace failed operating machines.

While in standby, spares may be selected for operation according to some rule known as the queue discipline. The different types of queue disciplines have been stated by several authors (e.g., Saaty [51], Gross and Harris [22]). The most familiar queue discipline is "first-come, first-served" (FCFS), or sometimes called "first-in, first-out" (FIFO), which means in our case, the spares are selected in the order of arrival. Other types of queue discipline include:

(1) "last-come, first-served" (LCFS), or "last-in, first-out" (LIFO);

(2) "random selection for service" (RSS), which means each spare has the same probability of being chosen for service;

(3) other priority rule.

We will provide the results of the simulation experiment using the suggested approach for choosing cut-off time and two modes of selection:

Selection I: Select a spare in a "random" fashion.

Selection II: Select the spare which is in the highest Erlang stage.

A spare at the latest Erlang stage is one which has aged the most as a warm standby. Upon selection, the machine is refurbished, and starts working in a state as good as new. This selection scheme tends to reduce the number of failed warm standby machines over a period of time, and thus may be desirable in practice.

The form of the Erlang distribution is E_K where K is the number of "stages". The Erlang distribution E_K has the negative exponential distribution when K = 1. The simulation run are performed on a M/M/R MRP and a $E_2/E_2/R$ MRP. We only choose a spare in a random fashion (Selection I) for the M/M/R case. The two modes of selection for spares are carried out for the $E_2/E_2/R$ case.

6.2.4 Comparative Analysis for the System Characteristics Between Exact, Approximate, and Simulation Results to the M/M/R Machine Repair Problem with Warm Standbys

The exact steady-state p.m.f. P_n obtained from birth and death equations and the approximate steady-state p.d.f. f(x) obtained from diffusion approximation to the M/M/R and the G/G/R machine repair problems with warm standbys have already been obtained in Chapter 2 and Chapter 3.

We provide a comparison for the expected number of failed machines in the system for the M/M/R MRP with warm standbys between the exact results, the approximate results obtained from diffusion approximation, and the simulation results.

We assume that at time $t = 0$, the system has just started operation with no failed machines. The simulation result is estimated by taking the simulation runs up to time $t = 1000$. Let

EXA = the exact result for the expected number of failed machines in the system,

DIF = the approximate result for the expected number of failed machines using diffusion approximation,

SIM = the simulation result for the expected number of failed machines including transients,

SIM_a = the simulation result for the expected number of failed machines excluding transients using cut-off times as shown in Table 6.4.

Exact results (EXA) and diffusion approximation results (DIF) obtained in previous works are provided by comparing them to the simulation results shown in Table 6.5.

We can see from Table 6.5 that for this example the selection of the cut-off time is not a significant factor to be incorporated in the simulation analysis.

Table 6.5 Comparisons of the expected number of failed machines between diffusion approximation and simulation results to the M/M/R MRP with warm standbys (M = 10, S = 5, simulation time = 1000).

$\alpha = 0.05$		R					
λ		1	3	5	6	8	10
	EXA	9.980	2.796	2.174	2.146	2.140	2.139
	DIF	9.962	2.924	2.289	2.257	2.248	2.248
0.2	SIM	9.974	2.694	2.197	2.307	2.354	2.314
	SIM_a	9.913	2.769	2.168	2.229	2.235	2.275
	EXA	12.500	7.397	4.471	4.107	3.949	3.937
	DIF	12.406	7.410	4.509	4.145	3.981	3.967
0.4	SIM	12.369	7.317	4.515	4.099	4.100	4.049
	SIM_a	12.385	7.357	4.360	4.267	3.992	3.917
	EXA	13.333	9.981	6.925	6.078	5.502	5.427
	DIF	13.182	9.962	6.934	6.090	5.512	5.433
0.6	SIM	13.302	9.961	6.957	5.854	5.534	5.447
	SIM_a	13.179	10.021	6.895	5.874	5.582	5.414
	EXA	13.750	11.248	8.776	7.781	6.790	6.586
	DIF	13.563	11.204	8.770	7.782	6.794	6.587
0.8	SIM	13.708	11.286	8.927	7.608	6.780	6.667
	SIM_a	13.670	11.149	8.713	7.720	6.716	6.541
	EXA	14.000	12.000	10.001	9.076	7.864	7.503
	DIF	13.790	11.930	9.983	9.069	7.865	7.504
1.0	SIM	13.847	11.853	9.868	8.971	7.891	7.372
	SIM_a	13.768	11.949	9.931	8.995	7.807	7.555

Table 6.5--Continued

α = 0.05		R					
λ		1	3	5	6	8	10
	EXA	14.167	12.500	10.833	10.022	8.760	8.255
	DIF	13.941	12.406	10.800	10.005	8.757	8.255
1.2	SIM	13.934	12.288	10.750	9.934	8.809	8.277
	SIM_a	13.949	12.449	10.816	9.762	8.739	8.353
	EXA	14.286	12.857	11.428	10.721	9.501	8.889
	DIF	14.050	12.742	11.380	10.692	9.492	8.888
1.4	SIM	14.055	12.756	11.233	10.719	9.495	8.820
	SIM_a	13.986	12.762	11.371	10.659	9.354	8.847
	EXA	14.375	13.125	11.875	11.253	10.112	9.433
	DIF	14.133	12.990	11.811	11.210	10.096	9.428
1.6	SIM	14.145	12.957	11.805	11.129	9.971	9.409
	SIM_a	14.113	12.976	11.805	11.196	10.049	9.409
	EXA	14.444	13.333	12.222	11.668	10.615	9.904
	DIF	14.198	13.182	12.143	11.612	10.590	9.894
1.8	SIM	14.141	13.152	12.094	11.490	10.473	9.894
	SIM_a	14.151	13.066	12.095	11.592	10.494	9.874
	EXA	14.500	13.500	12.500	12.000	11.033	10.314
	DIF	14.250	13.335	12.406	11.932	10.988	10.298
2.0	SIM	14.135	13.227	12.418	11.964	10.951	10.218
	SIM_a	14.162	13.365	12.475	11.925	10.933	10.308

6.2.5 Comparative Analysis for the System Characteristics Between Approximate Results and Simulation Results to the $E_2/E_2/R$ Machine Repair Problem with Warm Standbys

Let $C_M = \lambda^2 \sigma_M^2$, $C_S = \alpha^2 \sigma_S^2$, and $C_R = \mu^2 \sigma_R^2$, where C_M, C_S, and C_R are the square coefficients of variation of the succession of the uptimes of the operating machines, the uptimes of the spare machines, and the repair times, respectively. We also assume that at time $t = 0$, the system has just started operation with no failed machines. Recall that the two modes of selection are

(1) Selection I: select a spare in a "random" fashion;

(2) Selection II: select the spare which is in the "highest" Erlang stage.

For the numerical example, we choose the square coefficient of variation of the succession of the uptimes of the spare machines to be 0.2 (i.e., $C_S = 0.2$) and describe these uptimes by the E_K $(K = 5)$ distribution ($C_S = 1/K$), where K is the number of stages. Thus the highest Erlang stage is 5, next is 4, and so on.

The closed-form transient solution for the $E_2/E_2/R$ MRP with warm standbys has not yet been solved. However, a cut-off time for the $E_2/E_2/R$ MRP with warm standbys may be estimated approximately from the transient solution of the M/M/R case.

We first provide the simulation results using the two modes of selection but including the transients. Next, we provide the simulation results using either Selection I or Selection II, and exclude the transients using the cut-off time given in Table 6.4.

Let

$DIF_E \equiv$ the expected number of failed machines using diffusion approximation,

$SIM_I \equiv$ the simulation results of the expected number of failed machines using Selection I and including the transients,

$SIM_{II} \equiv$ the simulation results of the expected number of failed machines using Selection II and including the transients.

The diffusion approximation results (DIF_E) and the simulation results (SIM_I and SIM_{II}) for the expected number of failed machines to the $E_2/E_2/R$ MRP with warm standbys are shown in Table 6.6.

We now provide the results of the simulation analysis when excluding the transients.

Let

SIM_{IC} ≡ the simulation results of the expected number of failed machines using Selection I excluding the transients and using the cut-off time shown in Table 6.4,

SIM_{IIC} ≡ the simulation results of the expected number of failed machines using Selection II excluding the transients and using the cut-off time shown in Table 6.4.

The simulation results for Selection I and Selection II are shown in Table 6.7.

From an analysis of Table 6.6 and Table 6.7, it is evident that the inclusion of transients does not significantly affect the results. One sees from Table 3 and Table 4 that the simulation results excluding the transients and using cut-off time are closer to the diffusion approximation results than those including the transients. Further the selection mode is not of any particular consequence in determining the expected number of failed machines in the system. Finally, it is clear that diffusion theory provides very good approximation to the problem.

It should be noted however that had we selected a shorter simulation run, the transients would have normally affected the results. Also, it is conceivable that the use of other priority disciplines in the selection mode for spares may have resulted in different answers.

Table 6.6 Comparisons of the expected number of failed machines between diffusion approximation and simulation results to the $E_2/E_2/R$ MRP with warm standbys using <u>two modes of spare selection</u> ($M = 10$, $S = 5$, $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, simulation time = 1000).

$\alpha = 0.05$		R					
λ		1	3	5	6	8	10
0.2	DIF_E	9.998	2.449	2.160	2.154	2.154	2.154
	SIM_I	8.641	2.166	2.011	1.969	1.960	2.019
	SIM_{II}	8.678	2.322	1.868	1.961	2.003	1.974
0.4	DIF_E	12.491	7.418	4.263	4.039	3.989	3.989
	SIM_I	11.689	6.852	4.183	3.944	3.829	3.771
	SIM_{II}	11.623	7.126	4.196	3.983	3.863	3.781
0.6	DIF_E	13.305	9.998	6.798	5.920	5.525	5.507
	SIM_I	12.861	9.750	6.762	5.874	5.376	5.333
	SIM_{II}	12.707	9.500	6.785	6.051	5.279	5.372
0.8	DIF_E	13.703	11.248	8.755	7.664	6.725	6.633
	SIM_I	13.274	10.988	8.657	7.596	6.650	6.526
	SIM_{II}	13.327	10.950	8.396	7.532	6.621	6.529
1.0	DIF_E	13.937	11.995	10.000	9.026	7.740	7.501
	SIM_I	13.641	11.586	9.701	8.885	7.682	7.469
	SIM_{II}	13.635	11.738	9.775	8.896	7.624	7.455
1.2	DIF_E	14.091	12.491	10.832	10.004	8.634	8.210
	SIM_I	13.824	12.167	10.754	9.816	8.589	8.135
	SIM_{II}	13.722	12.223	10.665	9.830	8.585	8.141
1.4	DIF_E	14.200	12.842	11.426	10.714	9.403	8.816
	SIM_I	13.918	12.659	11.287	10.543	9.351	8.659
	SIM_{II}	13.950	12.562	11.230	10.549	9.381	8.865
1.6	DIF_E	14.281	13.104	11.871	11.249	10.045	9.345
	SIM_I	14.060	12.821	11.609	11.095	9.937	9.237
	SIM_{II}	14.046	12.912	11.699	11.090	9.892	9.388
1.8	DIF_E	14.344	13.305	12.216	11.664	10.573	9.812
	SIM_I	14.085	13.103	12.054	11.526	10.485	9.759
	SIM_{II}	14.164	13.078	11.956	11.588	10.384	9.836
2.0	DIF_E	14.395	13.466	12.491	11.995	11.006	10.227
	SIM_I	14.168	13.284	12.272	11.799	10.826	10.201
	SIM_{II}	14.191	13.252	12.267	11.888	10.911	10.174

Table 6.7 Simulation results with cut-off time using <u>two modes of spare selection</u> for the expected number of failed machines to the $E_2/E_2/R$ MRP with warm standbys ($M = 10$, $S = 5$, $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$, simulation Time = 1000).

$\alpha = 0.05$		R					
λ		1	3	5	6	8	10
0.2	SIM_{IC}	10.053	2.430	2.150	2.013	2.046	2.036
	SIM_{IIC}	10.197	2.305	2.094	2.100	2.111	2.024
0.4	SIM_{IC}	12.632	7.266	4.253	4.087	3.888	3.973
	SIM_{IIC}	12.447	7.356	4.236	4.030	3.990	3.847
0.6	SIM_{IC}	13.188	9.968	6.613	6.046	5.480	5.436
	SIM_{IIC}	13.193	9.810	6.776	5.996	5.525	5.452
0.8	SIM_{IC}	13.716	11.179	8.800	7.705	6.755	6.539
	SIM_{IIC}	13.690	11.207	8.805	7.650	6.818	6.580
1.0	SIM_{IC}	13.938	12.001	9.979	9.021	7.726	7.479
	SIM_{IIC}	13.953	11.960	9.801	9.067	7.726	7.576
1.2	SIM_{IC}	14.067	12.496	10.837	9.848	8.770	8.188
	SIM_{IIC}	14.028	12.510	10.765	9.915	8.685	8.294
1.4	SIM_{IC}	14.140	12.800	11.392	10.722	9.408	8.876
	SIM_{IIC}	14.185	12.783	11.427	10.786	9.449	8.888
1.6	SIM_{IC}	14.245	13.095	11.810	11.235	10.047	9.433
	SIM_{IIC}	14.234	13.037	11.814	11.201	10.085	9.452
1.8	SIM_{IC}	14.311	13.234	12.109	11.550	10.564	9.851
	SIM_{IIC}	14.333	13.270	12.186	11.625	10.570	9.903
2.0	SIM_{IC}	14.316	13.439	12.460	11.986	10.994	10.236
	SIM_{IIC}	14.358	13.393	12.439	11.936	10.972	10.251

6.2.6 Conclusion

Simulation provides a powerful tool to solve a complex machine repair problem with spares, incorporating a variety of selection modes for replacing failed operating machines. Theoretical transient results provide important decisions to determine cut-off time for termination of the transients in order to analyze simulation problem. In general, in a simulation analysis, the simulation results are affected by the choice of the cut-off time for transients particularly when the simulation run is short.

6.3 Sensitivity Analysis for the System Characteristics to the G/G/R Machine Repair Problem with Warm Standbys

In this section, we perform a sensitivity analysis for such system characteristics as machine availability (MA_G), and operative utilization (OU_G), for the G/G/R MRP with warm standbys when varying the square coefficients of variations C_M, C_S, and C_R.

Here we choose $M = 10$, $S = 5$, $\theta_\alpha = 0.05$ while varying the number of repairmen R ($3 \le R \le 8$) and θ ($0.2 \le \theta \le 2.0$). The results of MA_G, OU_G are given in Table 6.8 for two sets of values, namely,

(i) $C_M = 1.0$, $C_S = 1.0$, $C_R = 1.0$;

(ii) $C_M = 0.5$, $C_S = 0.2$, $C_R = 0.5$;

(iii) $C_M = 0.25$, $C_S = 0.1$, $C_R = 0.25$.

The set of values for which the square coefficients of variation are equal to one corresponds to the M/M/R MRP problem. One sees from Table 6.8 that for different values of the square coefficients of variation C_M, C_S, and C_R results for the system characteristics are not affected significantly. Therefore, the model for the M/M/R MRP provides

an excellent approximation to obtain such system characteristics as machine availability and operative utilization for the G/G/R MRP.

Table 6.8 Approximate of the machine availability and the operative utilization for the G/G/R MRP with warm standbys.

M = 10, S = 5		R							
		3		5		6		8	
θ_λ		MA_G	OU_G	MA_G	OU_G	MA_G	OU_G	MA_G	OU_G
	(i)	0.805	0.721	0.847	0.448	0.850	0.374	0.850	0.281
0.2	(ii)	0.837	0.710	0.856	0.431	0.856	0.359	0.856	0.269
	(iii)	0.851	0.713	0.857	0.429	0.857	0.357	0.857	0.268
	(i)	0.653	0.904	0.778	0.617	0.788	0.519	0.791	0.390
0.3	(ii)	0.676	0.936	0.790	0.616	0.794	0.515	0.794	0.386
	(iii)	0.690	0.962	0.793	0.619	0.794	0.516	0.794	0.387
	(i)	0.506	0.976	0.699	0.762	0.724	0.652	0.735	0.495
0.4	(ii)	0.505	0.995	0.716	0.783	0.731	0.662	0.734	0.499
	(iii)	0.502	1.000	0.724	0.799	0.731	0.670	0.732	0.503
	(i)	0.404	0.994	0.616	0.867	0.658	0.764	0.681	0.590
0.5	(ii)	0.401	1.000	0.631	0.903	0.668	0.788	0.680	0.600
	(iii)	0.400	1.000	0.641	0.931	0.672	0.805	0.676	0.607
	(i)	0.336	0.999	0.538	0.931	0.594	0.848	0.633	0.673
0.6	(ii)	0.333	1.000	0.547	0.965	0.605	0.881	0.632	0.687
	(iii)	0.333	1.000	0.552	0.986	0.613	0.905	0.629	0.696
	(i)	0.288	1.000	0.470	0.966	0.534	0.905	0.588	0.742
0.7	(ii)	0.286	1.000	0.474	0.989	0.545	0.940	0.590	0.760
	(iii)	0.286	1.000	0.476	0.998	0.553	0.963	0.589	0.769
	(i)	0.253	1.000	0.415	0.983	0.481	0.943	0.547	0.799
0.8	(ii)	0.250	1.000	0.416	0.997	0.489	0.972	0.552	0.820
	(iii)	0.250	1.000	0.417	1.000	0.495	0.989	0.554	0.830

Table 6.8--Continued

M = 10, S = 5		R							
		3		5		6		8	
θ_λ		MA_G	OU_G	MA_G	OU_G	MA_G	OU_G	MA_G	OU_G
	(i)	0.226	1.000	0.371	0.991	0.435	0.965	0.510	0.844
0.9	(ii)	0.222	1.000	0.370	0.999	0.440	0.988	0.517	0.868
	(iii)	0.222	1.000	0.370	1.000	0.443	0.997	0.522	0.880
	(i)	0.205	1.000	0.334	0.996	0.395	0.979	0.476	0.880
1.0	(ii)	0.200	1.000	0.333	1.000	0.398	0.995	0.484	0.905
	(iii)	0.200	1.000	0.333	1.000	0.400	0.999	0.491	0.920
	(i)	0.173	1.000	0.280	0.999	0.333	0.992	0.416	0.930
1.2	(ii)	0.167	1.000	0.278	1.000	0.333	0.999	0.424	0.954
	(iii)	0.167	1.000	0.278	1.000	0.333	1.000	0.431	0.971
	(i)	0.151	1.000	0.241	1.000	0.287	0.997	0.367	0.959
1.4	(ii)	0.144	1.000	0.238	1.000	0.286	1.000	0.373	0.979
	(iii)	0.143	1.000	0.238	1.000	0.286	1.000	0.378	0.991
	(i)	0.134	1.000	0.213	1.000	0.253	0.999	0.327	0.975
1.6	(ii)	0.126	1.000	0.209	1.000	0.250	1.000	0.330	0.991
	(iii)	0.125	1.000	0.208	1.000	0.250	1.000	0.333	0.998
	(i)	0.121	1.000	0.190	1.000	0.226	0.999	0.294	0.985
1.8	(ii)	0.113	1.000	0.186	1.000	0.222	1.000	0.295	0.996
	(iii)	0.111	1.000	0.185	1.000	0.222	1.000	0.296	0.999
	(i)	0.111	1.000	0.173	1.000	0.205	1.000	0.267	0.991
2.0	(ii)	0.102	1.000	0.167	1.000	0.200	1.000	0.266	0.998
	(iii)	0.100	1.000	0.167	1.000	0.200	1.000	0.267	1.000

CHAPTER 7

APPLICATION AND FUTURE RESEARCH

Our primary objective in this chapter is to provide an application to the M/M/R MRP with spares in which two types of repair rate are considered and to investigate possible extensions of the present work for future research.

7.1 Application

We study the availability characteristics of a system with M operating machines, S spares and R repairmen in the repair facility. Spares are considered to be either cold-standby, or warm-standby, or hot-standby. A standby component is called a "hot standby" if its failure rate is the same as an operating unit. The standby machine is referred to as "warm standby" when the failure rate is nonzero and is less than the failure rate of an operating unit, and the standby machine is referred to as "cold standby" when the failure rate is zero. Each repairman serves at the slow rate μ_1 until there are n ($n \geq R$) failed machines in the system, at which epoch he switches to the fast rate μ ($\mu_1 \leq \mu$). This increase in service rate often arises in real life problem to reduce the level of failed machines in the queue.

System failure is defined as the failure of at least one of the M operating machine (i.e., if more than S machines have failed). Our objective is to investigate the availability characteristics of the

M/M/R MRP with spares and with two mean repair rates under steady-state conditions. This problem should be distinguished from previous works in that: (i) it generalizes the MRP with spares for two types of repair rate, and (ii) it studies the availability characteristics of the machine repair problem with spares for two types of repair rate.

This particular model has been applied to study the availability characteristics of several unmanned air vehicles (UAV) while performing simultaneously surveillance and/or observation missions. The UAV's may be subject to failures and repairs and once recovered are catapulted back in the air by launching pads which act as repair facilities. Spare UAVs are used to improve the system availability.

7.1.1 Steady-State Solutions for a Variation of the M/M/R Machine Repair Model with Spares

Following the procedures given in Section 2.1.2 of Chapter 2, the general solutions of the M/M/R MRP with spares (either cold standbys, or warm standbys, or hot standbys) are obtained for two mean repair rates under steady-state conditions.

For this M/M/R model, the mean arrival rate λ_n is given by

$$(7.1) \qquad \lambda_n = \begin{cases} M\lambda + (S - n)\alpha, & \text{for } n = 0, 1, 2, \ldots, S, \\ (N - n)\lambda, & \text{for } n = S+1, S+2, \ldots, M+S-1 \\ 0, & \text{otherwise.} \end{cases}$$

The mean repair rate μ_n is given by

$$(7.2) \qquad \mu_n = \begin{cases} n\mu_1, & \text{for } n = 1, 2, \ldots, R-1, \\ R\mu & \text{for } n = R, R+1, \ldots, M+S=N, \\ 0, & \text{otherwise.} \end{cases}$$

In steady-state, let

P_0 = probability that no machines are broken down, and

P_n = probability that there are n failed machines in the system, where n = 1, ..., N.

The equilibrium equations for P_n are given by

(i) when $R \leq S$

(7.3a) $$(M\lambda+S\alpha)P_0 = \mu_1 P_1,$$

(7.3b) $$[M\lambda+(S-n)\alpha+n\mu]P_n = [M\lambda+(S-n+1)\alpha]P_{n-1} + (n+1)\mu_1 P_{n+1}, \quad 1 \leq n < R-1,$$

(7.3c) $$[M\lambda+(S-n)\alpha+n\mu]P_n = [M\lambda+(S-n+1)\alpha]P_{n-1} + R\mu P_{n+1}, \quad n = R-1,$$

(7.3d) $$[M\lambda+(S-n)\alpha+R\mu]P_n = [M\lambda+(S-n+1)\alpha]P_{n-1} + R\mu P_{n+1}, \quad R \leq n \leq S,$$

(7.3e) $$[(N-n)\lambda+R\mu]P_n = [(N-n+1)\lambda]P_{n-1} + R\mu P_{n+1}, \quad S < n < N,$$

(7.3f) $$\lambda P_{N-1} = R\mu P_N,$$

and (ii) when $R > S$

(7.4a) $$(M\lambda+S\alpha)P_0 = \mu_1 P_1$$

(7.4b) $$[M\lambda+(S-n)\alpha+n\mu]P_n = [M\lambda+(S-n+1)\alpha]P_{n-1} + (n+1)\mu_1 P_{n+1}, \quad 1 \leq n \leq S,$$

(7.4c) $$[(N-n)\lambda+n\mu]P_n = [(N-n+1)\lambda]P_{n-1} + (n+1)\mu_1 P_{n+1}, \quad S < n < R-1,$$

(7.4d) $$[(N-n)\lambda+n\mu]P_n = [(N-n+1)\lambda]P_{n-1} + R\mu P_{n+1}, \quad n = R-1,$$

(7.4e) $$[(N-n)\lambda+R\mu]P_n = [(N-n+1)\lambda]P_{n-1} + R\mu P_{n+1}, \quad R \leq n < N,$$

(7.4f) $$\lambda P_{N-1} = R\mu P_N.$$

Using the general birth and death results given by

$$P_n = \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} P_0,$$

we obtain the steady state solutions respectively

(i) when $R \le S$

(7.5a) $$P_n = \frac{1}{n!} \prod_{j=0}^{n-1} [M\rho_\lambda + (S - j)\rho_\alpha]P_0, \qquad 1 \le n < R,$$

(7.5b) $$P_n = \frac{(\mu_1/\mu)^{n-R+1}}{R!\ R^{n-R}} \prod_{j=0}^{n-1} [M\rho_\lambda + (S - j)\rho_\alpha]P_0, \qquad R \le n \le S,$$

(7.5c) $$P_n = \frac{(M-1)!\ \rho_\lambda^{n-S-1}\ (\mu_1/\mu)^{n-R+1}}{(N-n)!\ R!\ R^{n-R}} \prod_{j=0}^{S} [M\rho_\lambda + (S - j)\rho_\alpha]P_0, \qquad S < n \le N,$$

and (ii) when $R > S$

(7.6a) $$P_n = \frac{1}{n!} \prod_{j=0}^{n-1} [M\rho_\lambda + (S - j)\rho_\alpha]P_0, \qquad 1 \le n \le S,$$

(7.6b) $$P_n = \frac{(M-1)!\ \rho_\lambda^{n-S-1}}{(N-n)!\ n!} \prod_{j=0}^{S} [M\rho_\lambda + (S - j)\rho_\alpha]P_0, \qquad S < n < R,$$

(7.6c) $$P_n = \frac{(M-1)!\ \rho_\lambda^{n-S-1}\ (\mu_1/\mu)^{n-R+1}}{(N-n)!\ R!\ R^{n-R}} \prod_{j=0}^{S} [M\rho_\lambda + (S - j)\lambda_\alpha]P_0, \qquad R \le n \le N,$$

where

(7.7) $$\rho_\lambda = \frac{\lambda}{\mu_1} \quad \text{and} \quad \rho_\alpha = \frac{\alpha}{\mu_1}.$$

The results for the hot-standby model with two types of repair rate are obtained by setting $\alpha = \lambda$ in equation (7.5) (or equation (7.6)) given by

$$P_n = \frac{(M + S)!}{(M + S - n)!\ n!} \rho_\lambda^n P_0, \qquad 1 \le n < R,$$

(7.8)

$$P_n = \frac{(M + S)!\ (\mu_1/\mu)^{n-R+1}}{(M + S - n)!\ R!\ R^{n-R}} \rho_\lambda^n P_0, \quad R \le n \le M+S.$$

It should be noted that (i) when $\mu_1 = \mu$, and $0 < \alpha < \lambda$, equations (7.5) and (7.6) reduce to the existing results for the warm-standby case in equations (2.6) and (2.7), respectively, (ii) when $\mu_1 = \mu$, and $\alpha = 0$, equations (7.5) and (7.6) reduce to the existing results for the cold-standby case in the literature (Gross and Harris [22]), and (iii) when $\mu_1 = \mu$, $S = 0$, and $\alpha = 0$, equation (7.6) reduces to the existing results for the no-spare case in the literature (Feller [14]).

It should be noted that when $\mu_1 = \mu$ (i.e., $\theta_\lambda = \rho_\lambda$, $\theta_\alpha = \rho_\alpha$), equations 7.5 and equations 7.6 reduce to equations 2.6 and equations 2.7 in Chapter 2, respectively.

For both equations 7.5 and equations 7.6, P_0 can be solved from the normalizing equation

$$\sum_{n=0}^{N} P_n = 1.$$

7.1.2 Availability Characteristics of the Machine Repair Problem

System failure is defined as the failure of at least one of the M operating machine (i.e., if more than S machines have failed). The steady-state availability is the steady-state probability that at least M operating machines are functioning properly. The steady-state availability, say A, is given by

(7.9) $A = P(\text{"at least M operating machines are functioning properly"})$

$= P_0 + P_1 + \ldots + P_S.$

The availability of the system can be improved by providing sufficient standby spares. The availability analysis is carried out for the following three cases.

Case 1.a: Here we select $\alpha = \lambda = 6$ (hot-standby case), $\mu_1 = \mu = 12$, and vary the number of repairmen R ($1 \leq R \leq 10$);

Case 1.b: Here we select $\alpha = 0$ (cold-standby case), $\lambda = 6$, $\mu_1 = \mu = 12$, and vary the number of repairmen R ($1 \leq R \leq 10$);

Case 2.a: Here we select $\alpha = \lambda = 6$ (hot-standby case), $\mu_1 = \mu = 16$, and vary the number of repairmen R ($1 \leq R \leq 10$).

Case 2.b: Here we select $\alpha = 0$ (cold-standby case), $\lambda = 6$, $\mu_1 = \mu = 16$, and vary the number of repairmen R ($1 \leq R \leq 10$).

Case 3.a: Here we select $\alpha = \lambda = 6$ (hot-standby case), $\mu_1 = 12$, $\mu = 16$, and vary the number of repairmen R ($1 \leq R \leq 10$).

Case 3.b: Here we select $\alpha = 0$ (cold-standby case), $\lambda = 6$, $\mu_1 = 12$, $\mu = 16$, and vary the number of repairmen R ($1 \leq R \leq 10$).

Tables 7.1 through 7.12 show the number of standby spares that should be provided in order to maintain a minimum level of system availability, A, for various values of R. The results are displayed (i) for Case 1.a and Case 1.b in Tables 7.1 and 7.4 for M = 5 and M = 10, (ii) for Case 2.a and Case 2.b in Tables 7.5 and 7.8 for M = 5 and M = 10, and (iii) for Case 3.a and Case 3.b in Tables 7.9 and 7.12 for M = 5 and M = 10.

It should be noted that (i) if $M\lambda/R\mu \ll 1$, then the availability will approach unity as the number of spares increases, and (ii) if $M\lambda/R\mu > 1$, then the availability does not increase even if the system has an infinite number of spares.

Tables 7.1 through 7.12 show that (i) the availability increases with the number of spares, although in some cases the required availability may not be achieved even when providing an infinite number of spares, and (ii) the system availability improves when it has a fast repair rate. One sees for example from Table 7.4 that

(A) for $R = 5$, 23 standby spares are required to maintain an availability level of at least 0.85 but less than 0.9;

(B) by increasing the number of repairmen to $R = 6$, the number of spares is cut by more than half to 10 to maintain the availability at the same level as (A).

Table 7.1 Availability of the M/M/R model for <u>Case 1.a</u> (hot-standby case) when $\mu_1 = \mu = 12$, $M = 5$.

A \ R	1 — 2	3	4	5	6 — 10
0.6		3	3	3	3
0.65		4	3	3	3
0.7		7	3	3	3
0.75	N/A		4	4	4
0.8			5	4	4
0.85		N/A	6	5	4
0.9			14	6	5
0.95			N/A	8	6

* N/A means we cannot know how many spares provide.

Table 7.2 Availability of the M/M/R model for <u>Case 1.a</u> (hot-standby case) when $\mu_1 = \mu = 12$, $M = 10$.

A \ R	1 — 5	6	7	8	9	10
0.6		6	6	5	5	5
0.65		7	6	6	6	6
0.7		8	6	6	6	6
0.75	N/A	12	7	7	7	7
0.8			8	7	7	7
0.85			10	8	8	8
0.9		N/A		10	9	9
0.95			N/A	15	11	10

Table 7.3 Availability of the M/M/R model for <u>Case 1.b</u> (cold-standby case) when $\mu_1 = \mu = 12$, $M = 5$.

A \ R	1 — 2	3	4	5	6 — 10
0.6		3	3	2	2
0.65		3	3	3	3
0.7		4	3	3	3
0.75	N/A	5	3	3	3
0.8		5	4	4	4
0.85		7	4	4	4
0.9		8	5	5	4
0.95		12	7	6	5

Table 7.4 Availability of the M/M/R model for <u>Case 1.b</u> (cold-standby case) when $\mu_1 = \mu = 12$, $M = 10$.

A \ R	1 — 4	5	6	7	8	9 — 10
0.6		7	5	5	5	5
0.65		9	5	6	5	5
0.7		11	7	6	6	6
0.75	N/A	13	7	6	6	6
0.8		17	8	7	7	7
0.85		23	10	8	7	7
0.9		N/A	12	9	8	8
0.95			15	11	10	9

Table 7.5 Availability of the M/M/R model for Case 2.a (hot-standby case) when $\mu_1 = \mu = 16$, M = 5.

A \ R	1	2	3	4	5 — 10
0.6		5	2	2	2
0.65			2	2	2
0.7			3	2	2
0.75			3	3	3
0.8	N/A	N/A	4	3	3
0.85			5	4	3
0.9			13	4	4
0.95			N/A	6	5

Table 7.6 Availability of the M/M/R model for Case 2.a (hot-standby case) when $\mu_1 = \mu = 16$, M = 10.

A \ R	1 — 3	4	5	6	7	8 — 10
0.6		8	4	4	4	4
0.65			5	4	4	4
0.7			6	5	5	5
0.75			6	5	5	5
0.8	N/A	N/A	8	6	6	6
0.85			13	7	6	6
0.9				8	7	7
0.95			N/A	14	9	8

Table 7.7 Availability of the M/M/R model for <u>Case 2.b</u> (cold-standby case) when $\mu_1 = \mu = 16$, $M = 5$.

A \ R	1	2	3	4	5 — 10
0.6		3	2	2	2
0.65		4	2	2	2
0.7		5	2	2	2
0.75		6	3	3	3
0.8	N/A	7	3	3	3
0.85		9	4	3	3
0.9		13	5	4	4
0.95		21	6	5	4

Table 7.8 Availability of the M/M/R model for <u>Case 2.b</u> (cold-standby case) when $\mu_1 = \mu = 16$, $M = 10$.

A \ R	1 — 3	4	5	6	7 — 10
0.6		5	4	4	4
0.65		6	4	4	4
0.7		7	5	5	4
0.75		9	5	5	5
0.8	N/A	11	6	6	5
0.85		13	7	6	6
0.9		17	8	7	6
0.95		26	11	8	7

Table 7.9 Availability of the M/M/R model for <u>Case 3.a</u> (hot-standby case) when $\mu_1 = 12$, $\mu = 16$, $M = 5$.

A \ R	1	2	3	4	5	6 — 10
0.6		5	2	3	3	3
0.65			3	3	3	3
0.7			3	3	3	3
0.75			4	3	3	3
0.8	N/A	N/A	5	4	4	3
0.85			13	4	4	4
0.9				5	5	5
0.95			N/A	7	6	6

Table 7.10 Availability of the M/M/R model for Case 3.a (hot-standby case) when $\mu_1 = 12$, $\mu = 16$, $M = 10$.

A \ R	1 — 3	4	5	6	7	8	9	10
0.6	N/A	8	5	5	5	5	5	5
0.65		N/A	5	5	5	6	6	6
0.7			6	6	6	6	6	6
0.75			7	6	6	6	7	7
0.8			9	7	6	7	7	7
0.85			14	8	7	7	8	8
0.9			N/A	9	8	8	8	8
0.95				15	10	9	9	9

Table 7.11 Availability of the M/M/R model for Case 3.b (cold-standby case) when $\mu_1 = 12$, $\mu = 16$, $M = 5$.

A \ R	1	2	3	4 — 5	6 — 10
0.6	N/A	3	2	2	2
0.65		4	2	3	3
0.7		5	3	3	3
0.75		6	3	3	3
0.8		7	3	3	4
0.85		10	4	4	4
0.9		13	5	4	4
0.95		21	6	5	5

Table 7.12 Availability of the M/M/R model for Case 3.b (cold-standby case) when $\mu_1 = 12$, $\mu = 16$, M = 10.

A \ R	1 — 3	4	5	6	7	8	9	10
0.6		6	5	5	5	5	5	5
0.65		7	5	5	5	5	5	5
0.7		8	5	5	5	6	6	6
0.75		9	6	6	6	6	6	6
0.8	N/A	11	7	6	6	6	6	7
0.85		14	8	7	7	7	7	7
0.9		18	9	7	7	7	8	8
0.95		26	11	9	8	8	8	9

7.2 Future Research

7.2.1 Closed-Form Transient Solutions f(x;t) for the G/G/R Machine Repair Problem with Warm Standbys

In many practical problems, it is often useful to have an approximate transient solution to describe the behavior of queueing systems. Since many queueing processes never reach a steady-state, time-dependent solutions of queueing models are of great practical importance.

Approximate methods (e.g., numerical integration methods, and randomization procedure) to find the transient solution for the M/M/R MRP with cold standbys have appeared as an exercise in Gross and Harris's textbook [22] (page 454, ex. 7.12) for which no solution methodology is provided. Wang and Sivazlian [68] have recently provided the closed-form transient solution to the M/M/R MRP with warm standbys using transform techniques. For a more detailed solution methodology one should refer to Wang and Sivazlian [68].

The transient solution $f(x;t)$ of the Fokker-Planck equation subject to appropriate boundary condition for the G/G/R MRP with warm standbys is a problem which is open for future investigation. It should be noted that the diffusion parameters of this problem are not constant: they are linear functions of x. However, the solution of the Fokker-Planck equation with constant diffusion parameters for certain boundary conditions has been obtained by several authors including Sweet and Hardin [59], Cox and Miller [11], Karlin and Taylor [32], Newell [47], Reiser and Kobayashi [50], and Kobayashi [42]. Several different methods have been used to solve the Fokker-Planck equation including: the method of separation of variables [59], the method of eigenfunction expansion [42], and the Laplace transform method [14].

7.2.2 Steady-state Analysis of the G/G/R Machine Repair Problem with Warm Standbys Using Instantaneous Return Process

Another problem of particular interest for future research is to use the idea of the instantaneous return process to analyze the same model as in Chapter 3. The instantaneous return model has been investigated by Feller [14], Bharucha-Reid [7], Gelenbe [19], Kimura [35], and Kimura and Takahashi [36, 37].

Recall that the process $\{N(t), t \geq 0\}$ approximating the total number of failed machines in the G/G/R queue at time t takes values on the interval [0, N]. There are two boundaries which can be applied to the queueing model; the reflecting barrier (Chapter 1) and the elementary return boundary (Feller [14], Bharucha-Reid [7]). In this study, we treat the boundary 0 as an "elementary return" boundary. Roughly speaking, when the process $\{N(t), t \geq 0\}$ reaches the boundary 0, it stays there for an exponentially distributed time length and then it

returns to the interval (0, N] (Feller [14]). Following arrivals to the empty queue, the number of failed machines in the queue jumps instantaneously to x ($0 < x \leq N$), where x is the jump amounts for failed machines from 0. A jump amount corresponds to the first failed machine arriving in the queue after the queue becomes empty.

The p.d.f. f(x;t) satisfies the Fokker-Planck equation with diffusion parameters $A_{1i}(x)$ and $A_{2i}(x)$ for $0 < x \leq N$. Let $\pi_0(t)$ be the sojourn probability at x = 0 at time t: this represents the probability that the queue is empty. Let q(x) be the p.d.f. of the jump amounts from zero to x. In the instantaneous return model, the steady-state p.d.f. f(x) satisfies the equation given by (Kimura and Takahashi [36, 37]):

$$- \frac{d}{dx} [A_{1i}(x) f_i(x)] + \frac{1}{2} \frac{d^2}{dx^2} [A_{2i}(x) f_i(x)] = - \lambda_x \pi_0 q(x),$$

where π_0 is the stationary p.m.f. at x = 0, representing the probability that the queue is empty. The boundary condition is given by

$$\lambda_x\pi_0 = \{ - [A_{11}(x)f_1(x)] + \frac{1}{2} \frac{d}{dx} [A_{21}(x)f_1(x)]\}\Big|_{x=R}$$

$$- \{ - [A_{11}(x)f_1(x)] + \frac{1}{2} \frac{d}{dx} [A_{21}(x)f_1(x)]\}\Big|_{x=0}$$

$$+ \{ - [A_{12}(x)f_2(x)] + \frac{1}{2} \frac{d}{dx} [A_{22}(x)f_2(x)]\}\Big|_{x=S}$$

$$- \{ - [A_{12}(x)f_2(x)] + \frac{1}{2} \frac{d}{dx} [A_{22}(x)f_2(x)]\}\Big|_{x=R}$$

$$+ \{ - [A_{13}(x)f_3(x)] + \frac{1}{2} \frac{d}{dx} [A_{23}(x)f_3(x)]\}\Big|_{x=N}$$

$$- \{ - [A_{13}(x)f_3(x)] + \frac{1}{2} \frac{d}{dx} [A_{23}(x)f_3(x)]\}\Big|_{x=S}.$$

In the special case when the queue is empty; an arrival to the empty queue corresponds to an instantaneous jump from 0 to 1, therefore we must let $q(x)$ be the Dirac delta function, namely, $q(x) = \delta(x - 1)$. The steady-state p.d.f. $f(x)$ satisfies the equation given by (Gelenbe [19]):

$$- \frac{d}{dx} [A_{1i}(x)f_i(x)] + \frac{1}{2} \frac{d^2}{dx^2} [A_{2i}(x)f_i(x)] = - \lambda_0\pi_0\delta(x - 1),$$

and boundary condition

$$\lambda_0\pi_0 = \{ - [A_{11}(x)f_1(x)] + \frac{1}{2} \frac{d}{dx} [A_{21}(x)f_1(x)]\}\Big|_{x=R}$$

$$- \{ - [A_{11}(x)f_1(x)] + \frac{1}{2} \frac{d}{dx} [A_{21}(x)f_1(x)]\}\Big|_{x=0}$$

$$+ \{ - [A_{12}(x)f_2(x)] + \frac{1}{2} \frac{d}{dx} [A_{22}(x)f_2(x)]\}\Big|_{x=S}$$

$$- \{ - [A_{12}(x)f_2(x)] + \frac{1}{2} \frac{d}{dx} [A_{22}(x)f_2(x)]\}\Big|_{x=R}$$

$$+ \{ - [A_{13}(x)f_3(x)] + \frac{1}{2} \frac{d}{dx} [A_{23}(x)f_3(x)]\}\Big|_{x=N}$$

$$- \{ - [A_{13}(x)f_3(x)] + \frac{1}{2} \frac{d}{dx} [A_{23}(x)f_3(x)]\}\Big|_{x=S}.$$

Steady-state p.d.f. f(x) for the G/G/R MRP with warm standbys using the instantaneous return process is thus a solution to the above equations.

7.2.3 Diffusion Approximation to the G/G/R Machine Repair Problem with Priority Classes of Warm Standby Spares

In Chapter 6, we considered two modes of selection of available spares for replacing failed operating machines. The simulation results showed that the particular selection mode chosen was not a significant factor. However, such factors have not been incorporated in the diffusion approximation models. For example, it is possible to consider a problem where we assume the S spares are divided into S priority classes; a class j has a higher priority than class i, if and only if $i < j$. Thus, class S has the highest priority, and class 1 has the lowest priority. In each class we assume that spares start to replace

failed operating machines in order of arrival. The system characteristics of the G/G/R MRP with warm standby spares where spares have priority classes are open for future study.

7.2.4 Cost Analysis of the M/M/R Machine Repair Problem with Two Types of Failure Modes and Two Repair Policies

The problem in which two repair policies are considered for the M/M/R MRP where machines are subject to two types of failure modes is another area for future study. In this problem, policy I consists in having failed machines with higher priority of repair always start service prior to those failed machines with lower priority, but this priority is non-preemptive so repairs are never interrupted by a failed machine with higher priority. In policy II the two failure modes have equal probability of repair, that is, the service is according to FCFS. This class of problem is more general than the work of Elsayed [12] who considers the single-server case. General steady-state solutions for this model may be obtained from the birth and death equations. The steady-state expected cost of downtime losses and repairs per machine per unit time may be developed in order to determine the optimal number of machines and repairmen simultaneously for each policy. Under the optimal operating conditions, several system characteristics can be evaluated for the two types of failure modes. Depending on the values of the input and cost parameters, a decision can then be made for selecting one particular policy over the other.

APPENDIX A

VALIDITY OF EQUATIONS 2.8b AND 2.9b AT x = 0, AND VALIDITY OF EQUATIONS 2.8d AND 2.9d AT x = N

In the diffusion model of the M/M/R system, we show that

(i) equation 2.8b for $0 < x < R$ reduces to equation 2.8a at $x = 0$;

(ii) equation 2.8d for $S < x < N$ reduces to equation 2.8e at $x = N$;

(iii) equation 2.9b for $0 < x \leq S$ reduces to equation 2.9a at $x = 0$;

(iv) equation 2.9d for $R < x < N$ reduces to equation 2.9e at $x = N$;

proof: (i) Equation 2.8b is given by

$$\text{(A.1)} \quad \frac{\partial f(x;t)}{\partial t} = - [M\lambda + (S - x)\alpha + x\mu]f(x;t) + (x + 1)\mu\, f(x+1;t)$$
$$+ [M\lambda + (S - x + 1)\alpha]f(x-1;t).$$

In the right hand side of equation A.1, using Taylor series expansion about x, retaining only terms of the first- and second- order, and rearranging, we obtain

$$\text{(A.2)} \quad \frac{\partial f(x;t)}{\partial t} = (\alpha + \mu)f(x;t) - [M\lambda + (S - x + 1)\alpha - (x + 1)\mu]f'(x;t)$$
$$+ \frac{1}{2} [M\lambda + (S - x + 1)\alpha + (x + 1)\mu]f''(x;t).$$

Substituting $x = 0$ in equation A.2 and simplifying, we obtain

$$(A.3)\quad \frac{\partial f(0;t)}{\partial t} = (M\lambda + S\alpha)[\frac{1}{2} f''(0;t) - f'(0;t)]$$

$$+ \alpha [\frac{1}{2} f''(0;t) - f'(0;t) + f(0;t)]$$

$$+ \mu [\frac{1}{2} f''(0;t) + f'(0;t) + f(0;t)].$$

Substituting the following expressions

$$(A.4a)\quad f(0;t) - f'(0;t) + \frac{1}{2} f''(0;t) = 0,$$

and

$$(A.4b)\quad f(0;t) + f'(0;t) + \frac{1}{2} f''(0;t) = f(1;t),$$

in equation A.3, we obtain

$$\frac{\partial f(0;t)}{\partial t} = - (M\lambda + S\alpha)f(0;t) + \mu\, f(1;t),$$

which is the same as equation 2.8a.

(ii) Equation 2.8d is given by

$$(A.5)\quad \frac{\partial f(x;t)}{\partial t} = - [(N - x)\lambda + R\mu]f(x;t) + R\mu f(x+1;t)$$

$$+ [(N - x + 1)\lambda]f(x-1;t).$$

Again, we apply a Taylor series about x in equation A.5, retain only terms of the first- and second- order, and rearrange. Then we obtain

$$(A.6)\quad \frac{\partial f(x;t)}{\partial t} = \lambda f(x;t) + [R\mu - (N - x + 1)\lambda]f'(x;t)$$

$$+ \frac{1}{2} [R\mu + (N - x + 1)\lambda]f''(x;t).$$

Substituting x = N in equation A.6 and rearranging, we obtain

$$\text{(A.7)}\quad \frac{\partial f(N;t)}{\partial t} = \lambda\left[f(N;t) - f'(N;t) + \frac{1}{2} f''(N;t)\right] + R\mu\left[f'(N;t) + \frac{1}{2} f''(N;t)\right].$$

Substituting the following expressions

$$\text{(A.8a)}\quad f(N;t) - f'(N;t) + \frac{1}{2} f''(N;t) = f(N-1;t),$$

and

$$\text{(A.8b)}\quad f(N;t) + f'(N;t) + \frac{1}{2} f''(N;t) = 0,$$

in equation A.7, we obtain

$$\frac{\partial f(N;t)}{\partial t} = - R\mu\, f(N;t) + \lambda\, f(N-1;t)$$

which is the same as the equation 2.8e.

The (iii) and (iv) assertions follow similarly. ∎

APPENDIX B

THE CONDITIONAL EXPECTATION AND VARIANCE OF N(t+dt) - N(t), GIVEN N(t) = n FOR THE M/M/R SYSTEM

For the M/M/R system, during the time interval (t, t+dt), we show that:

(i) for $0 \le n < R$

$$E[N(t+dt) - N(t) | N(t) = n] = [M\lambda + (S - n)\alpha - n\mu]dt + o(dt),$$

$$Var[N(t+dt) - N(t) | N(t) = n] = [M\lambda + (S - n)\alpha + n\mu]dt + o(dt),$$

(ii) for $R \le n \le S$

$$E[N(t+dt) - N(t) | N(t) = n] = [M\lambda + (S - n)\alpha - R\mu]dt + o(dt),$$

$$Var[N(t+dt) - N(t) | N(t) = n] = [M\lambda + (S - n)\alpha + R\mu]dt + o(dt),$$

(iii) for $S < n \le N$

$$E[N(t+dt) - N(t) | N(t) = n] = [(N - n)\lambda - R\mu]dt + o(dt),$$

$$Var[N(t+dt) - N(t) | N(t) = n] = [(N - n)\lambda + R\mu]dt + o(dt).$$

Proof: For $0 \le n < R$, using equations (2.34) and (2.36), we obtain

(B.1) $$E[N(t+dt) - N(t) | N(t) = n]$$
$$= P\{N(t+dt) - N(t) = 1 | N(t) = n\}$$
$$+ (-1)\, P\{N(t+dt) - N(t) = -1 | N(t) = n\}$$
$$= [M\lambda + (S - n)\alpha - n\mu]dt + o(dt),$$

and

(B.2) $$E[\{N(t+dt) - N(t)\}^2 | N(t) = n]$$
$$= P\{N(t+dt) - N(t) = 1 | N(t) = n\}$$
$$+ (-1)^2\, P\{N(t+dt) - N(t) = -1 | N(t) = n\}$$
$$= [M\lambda + (S - n)\alpha + n\mu]dt + o(dt).$$

From equations (B.1) and (B.2), we obtain

$$\text{(B.3)} \quad \begin{aligned} &\mathrm{Var}[N(t+dt) - N(t) \mid N(t) = n] \\ &= E[\{N(t+dt) - N(t)\}^2 \mid N(t) = n] \\ &\quad - \{E[N(t+dt) - N(t) \mid N(t) = n]\}^2 \\ &= [M\lambda + (S - n)\alpha + n\mu]dt + o(dt). \end{aligned}$$

Using equations (2.37) and (2.39) when $R \le n \le S$, and equations (2.40) and (2.42) when $S < n \le N$, the (ii) and (iii) assertions follows similarly. ∎

APPENDIX C

PROOF OF THE EQUATIONS 2.52 AND 2.53

For the M/M/R system, during the time interval (t, t+dt), we show that:

$$(2.52)\quad E[N(t+dt) - N(t) \mid N(t) = n] = E[M(t+dt) - M(t) \mid N(t) = n] + E[S(t+dt) - S(t) \mid N(t) = n] - E[R(t+dt) - R(t) \mid N(t) = n],$$

$$(2.53)\quad Var[N(t+dt) - N(t) \mid N(t) = n] = Var[M(t+dt) - M(t) \mid N(t) = n] + Var[S(t+dt) - S(t) \mid N(t) = n] + Var[R(t+dt) - R(t) \mid N(t) = n].$$

Proof: For $0 \le n < R$, using the results in Table (2.9), we obtain

$$(C.1)\quad E[M(t+dt) - M(t) \mid N(t) = n] = P(M(t+dt) - M(t) = 1 \mid N(t) = n] = M\lambda dt + o(dt),$$

$$(C.2)\quad E[S(t+dt) - S(t) \mid N(t) = n] = P(S(t+dt) - S(t) = 1 \mid N(t) = n] = (S - n)\alpha dt + o(dt),$$

$$(C.3)\quad E[R(t+dt) - R(t) \mid N(t) = n] = P(R(t+dt) - R(t) = 1 \mid N(t) = n] = n\mu dt + o(dt).$$

From Appendix B, for $0 \le n < R$ we have

$$(C.4)\quad E[N(t+dt) - N(t) \mid N(t) = n] = [M\lambda + (S - n)\alpha - n\mu]dt + o(dt).$$

From equations (C.1) through (C.4), for $0 \le n < R$ we obtain

$$E[N(t+dt) - N(t) | N(t) = n] = E[M(t+dt) - M(t) | N(t) = n]$$
$$+ E[S(t+dt) - S(t) | N(t) = n]$$
$$- E[R(t+dt) - R(t) | N(t) = n].$$

The equation 2.52 assertion for the other case $R \le n \le S$, and $S < n \le N$ follows similarly.

For $0 \le n < R$, using the results in Table (2.9), we obtain

(C.5) $E[\{M(t+dt) - M(t)\}^2 | N(t) = n] = P\{M(t+dt) - M(t) = 1 | N(t) = n]$
$= M\lambda dt + o(dt),$

(C.6) $E[\{S(t+dt) - S(t)\}^2 | N(t) = n] = P\{S(t+dt) - S(t) = 1 | N(t) = n]$
$= (S - n)\alpha dt + o(dt),$

(C.7) $E[\{R(t+dt) - R(t)\}^2 | N(t) = n] = P\{R(t+dt) - R(t) = 1 | N(t) = n]$
$= n\mu dt + o(dt).$

Using equations (C.1) and (C.5), we obtain

(C.8) $\mathrm{Var}[M(t+dt) - M(t) | N(t) = n]$
$= E[\{M(t+dt) - M(t)\}^2 | N(t) = n]$
$- \{E[M(t+dt) - M(t) | N(t) = n]\}^2$
$= [M\lambda + (S - n)\alpha + n\mu]dt + o(dt).$

Likewise, using equations (C.2) and (C.6), we obtain

(C.9) $\mathrm{Var}[S(t+dt) - S(t) | N(t) = n] = (S - n)\alpha dt + o(dt).$

Using equations (C.3) and (C.7), we obtain

(C.10) $\mathrm{Var}[R(t+dt) - R(t) | N(t) = n] = n\mu dt + o(dt).$

From Appendix B, for $0 \le n < R$ we have

(C.11) $\mathrm{Var}[N(t+dt) - N(t) | N(t) = n] = [M\lambda + (S - n)\alpha + n\mu]dt + o(dt).$

From equations (C.8) through (C.11), for $0 \le n < R$ we obtain

$$
\begin{aligned}
\mathrm{Var}[N(t+dt) - N(t) \mid N(t) = n] &= \mathrm{Var}[M(t+dt) - M(t) \mid N(t) = n] \\
&\quad + \mathrm{Var}[S(t+dt) - S(t) \mid N(t) = n] \\
&\quad + \mathrm{Var}[R(t+dt) - R(t) \mid N(t) = n].
\end{aligned}
$$

The equation 2.53 assertion for the other case $R \le n \le S$, and $S < n \le N$ follows similarly. ∎

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